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On Fixed Point Theorem of Self Map in Fuzzy Metric Spaces

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ABSTRACT: In this paper, we prove a fixed point theorem for a self map in fuzzy metric space where the map satisfies a different condition.

KEYWORDS: t – norm, fuzzy metric space, cauchy sequence, complete fuzzy metric space.

I. INTRODUCTION

The fundamental work for the fuzzy theory was first given by Zadeh [9] in 1965, who introduced the concept of fuzzy set. Kramosil and Michalek [4] developed the fuzzy metric space and later George and Veeramani [1] modified the notion of fuzzy metric spaces by introducing the concept of continuous t – norm. Many researchers have enormously developed the theory by defining different concepts and blending many properties. Fuzzy set theory has its significance in various fields such as communication, gaming, signal processing, modelling theory, image processing, etc.

The purpose of this work is to prove the existence of a fixed point of a self map in fuzzy metric space (in the context of George and Veeramani [1]) by using certain condition on the mapping.

II. PRELIMINARIES

Definition 1.1([7]): A binary operation $*$: [0, 1] × [0, 1] → [0, 1] is a continuous t – norm if ([0, 1], $*$) is an abelian topological monoid with unit 1 such that $x * y \leq z * w$ whenever $x \leq z$ and $y \leq w$ (where $x, y, z, w \in [0, 1]$).

Definition 1.2([1]): let X be any non – empty set, $*$ be a continuous t – norm, M is a fuzzy set on X × X × (0, ∞) satisfying

i. $M(x, y, t) > 0$
ii. $M(x, y, t) = 1 \Leftrightarrow x = y$
iii. $M(x, y, t) = M(y, x, t)$
iv. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
v. $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous where $x, y, z \in X$, $s, t > 0$

Here, $M(x, y, t)$ represents the degree of nearness between x, y with respect to t.
Then, the 3- tuple (X, M, $*$) is called a fuzzy metric space.

Definition 1.3([1]): A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is said to converge to $x \in X \Leftrightarrow M(x_n, x, t) \to 1$ as $n \to \infty$.

Definition 1.4 ([1]): Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, $*$). If for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that

$$M(x_n, x_m, t) > 1 - \varepsilon \text{ for all } n, m \geq n_0$$

then the sequence $\{x_n\}$ is said to be a cauchy sequence in X.

Definition 1.5 ([1]): A fuzzy metric space is complete if every cauchy sequence is complete.

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III. MAIN RESULTS

Theorem 2.1: Let (X, M, $*$) be a complete fuzzy metric space where $*$ is a continuous t – norm defined by $a * b = \min\{a, b\}$. Let T be a self map on X which is continuous and satisfying the following condition

(2.1.1) $\quad M(Tx, Ty, t) * M(x, Ty, t) * M(Tx, y, t) \geq M(x, Tx, t) * M(y, Ty, t) * M(x, y, t)$

where $x, y \in X$ and $x \neq y$, then T has a fixed point in X.

In addition, if the fuzzy metric $M(x, y, t)$ satisfies the condition

(2.1.2) $\quad \lim_{t\to\infty} M(x, y, t) = 1$

Then, T has a unique fixed point in X.

Proof:let $\{x_n\}$be a sequence in X such that $Tx_n = x_{n+1}$.

If $x_n = x_{n+1}$, then $x_n = Tx_n$ which implies that x_n is a fixed point of T.

Let $x_n \neq x_{n+1}$

- To prove that $\{x_n\}$ is a cauchy sequence in X:

Put $x = x_{n-1}$ and $y = x_n$ in (2.1.1), we get

$$M(Tx_{n-1}, Tx_n, t) * M(x_{n-1}, Tx_n, t) * M(Tx_{n-1}, x_n, t) \geq M(x_{n-1}, Tx_{n-1}, t) * M(x_n, Tx_n, t) * M(x_{n-1}, x_n, t)$$

$$M(x_n, x_{n+1}, t) * M(x_{n-1}, x_{n+1}, t) * M(x_n, x_n, t) \geq M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t) * M(x_{n-1}, x_n, t)$$

$$M(x_n, x_{n+1}, t) * M(x_{n-1}, x_{n+1}, t) \geq M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t) * M(x_{n-1}, x_n, t)$$

$$M(x_n, x_{n+1}, t) * M(x_{n-1}, x_{n+1}, t) \geq M(x_{n-1}, x_n, t) * M(x_n, x_{n+1}, t) \quad (\because a * a = \min\{a,a\})$$

$$M(x_{n-1}, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$$

(2.1.3) $$M(x_{n+1}, x_{n-1}, t) \geq M(x_n, x_{n-1}, t) \quad \text{for all } n$$

Put $x = x_{n-1}$ and $y = x_{n-2}$in (2.1.1)

$$M(Tx_{n-1},Tx_{n-2},t) * M(x_{n-1},Tx_{n-2},t) * M(Tx_{n-1},x_{n-2}, t) \geq M(x_{n-1},Tx_{n-1},t) * M(x_{n-2},Tx_{n-2},t) * M(x_{n-1},x_{n-2},t)$$

$$M(x_n, x_{n-1},t) * M(x_{n-1}, x_{n-1},t) * M(x_n, x_{n-2}, t) \geq M(x_{n-1}, x_n, t) * M(x_{n-2}, x_{n-1}, t) * M(x_{n-1}, x_{n-2},t)$$

$$M(x_n, x_{n-1},t) * M(x_n, x_{n-2}, t) \geq M(x_{n-1}, x_n,t) * M(x_{n-2}, x_{n-1},t) \quad (\because a * a = \min\{a,a\})$$

$$M(x_n, x_{n-2}, t) \geq M(x_{n-2}, x_{n-1}, t)$$

(2.1.4) $$M(x_n, x_{n-2}, t) \geq M(x_{n-1}, x_{n-2}, t) \quad \text{for all } n$$

Suppose let us assume that $\{x_n\}$ is not a cauchy sequence

then, for $0 < \varepsilon < 1$, $t > 0$, there exists sequences $\{x_{n_0}\}$ and $\{x_{m_0}\}$ where $n_0, m_0 \geq n$ & $n_0, m_0 \in N$ $(n_0 > m_0)$ such that

(2.1.5) $\quad M(x_{n_0}, x_{m_0}, t) \leq 1 - \varepsilon$

(2.1.6) $\quad M(x_{n_0-1}, x_{m_0-1}, t) > 1 - \varepsilon \;,\; M(x_{n_0-1}, x_{m_0}, t) > 1 - \varepsilon$

Consider

$$\begin{aligned}
1 - \varepsilon &\geq M(x_{n_0}, x_{m_0}, t) \\
&\geq M(x_{n_0}, x_{n_0-1}, \tfrac{t}{2}) * M(x_{n_0-1}, x_{m_0}, \tfrac{t}{2}) \\
&\geq M(x_{n_0}, x_{n_0-2}, \tfrac{t}{4}) * M(x_{n_0-2}, x_{n_0-1}, \tfrac{t}{4}) * M(x_{n_0-1}, x_{m_0}, \tfrac{t}{2}) \\
&\geq M(x_{n_0-1}, x_{n_0-2}, \tfrac{t}{4}) * M(x_{n_0-2}, x_{n_0-1}, \tfrac{t}{4}) * M(x_{n_0-1}, x_{m_0}, \tfrac{t}{2}) && (\because \text{from } (2.1.4)) \\
&\geq M(x_{n_0-1}, x_{n_0-2}, \tfrac{t}{4}) * M(x_{n_0-1}, x_{m_0}, \tfrac{t}{2}) \\
&\geq M(x_{n_0-2}, x_{n_0-1}, \tfrac{t}{4}) * M(x_{n_0-1}, x_{m_0}, \tfrac{t}{2}) \\
&\geq M(x_{n_0-1}, x_{n_0-1}, \tfrac{t}{4}) * M(x_{n_0-1}, x_{m_0}, \tfrac{t}{2}) && (\because \text{from } (2.1.3)) \\
&\geq 1 * M(x_{n_0-1}, x_{m_0}, \tfrac{t}{2}) \\
&= M(x_{n_0-1}, x_{m_0}, \tfrac{t}{2}) \\
&> 1 - \varepsilon && (\because \text{from } (2.1.6))
\end{aligned}$$

which is a contradiction.

$\therefore \{x_n\}$ is a cauchy sequence.

Since X is complete, there exists an element $z \in X$ such that $\lim_{n\to\infty} x_n = z$

i.e., $M(x_n, z, t) = 1$ as $n \to \infty$

- To prove that limit z is unique:

Let if possible

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$\lim_{n\to\infty} x_n = u$ for some $u \in$ X where $u \neq z$

then

$$\text{M}(u, z, t) \geq \text{M}(u, x_n, \tfrac{t}{2}) * \text{M}(x_n, z, \tfrac{t}{2})$$

taking limit $n \to \infty$ on both sides

$$\text{M}(u, z, t) \geq \text{M}(u, u, \tfrac{t}{2}) * \text{M}(z, z, \tfrac{t}{2})$$

$$\text{M}(u, z, t) \geq 1 * 1$$

$\text{M}(u, z, t) \geq 1$, a contradiction.

Therefore, limit z is unique.

- To show that z is a fixed point of T:

(2.1.7) Since T is continuous, $x_n \to z \Rightarrow \text{T}x_n \to \text{T}z$

Consider (2.1.3)

$$\text{M}(x_{n+1}, x_{n-1}, t) \geq \text{M}(x_n, x_{n-1}, t)$$

$$\text{M}(\text{T}x_n, x_{n-1}, t) \geq \text{M}(x_n, x_{n-1}, t)$$

Letting $n \to \infty$, we get

$\text{M}(\text{T}z, z, t) \geq \text{M}(z, z, t)$ (since $\{x_n\}$ is a cauchy sequence)

$$\text{M}(\text{T}z, z, t) \geq 1$$

$$\therefore \text{M}(\text{T}z, z, t) = 1 \quad \Rightarrow \text{T}z = z$$

i.e., z is a fixed point of T.

- To prove Uniqueness of fixed point:

Suppose that w is another fixed point of T

i.e., T$w = w$ where $w \neq z$

Now we show that $w = z$

Consider

$$\text{M}(z, w, t) < 1$$

$$\begin{aligned}
\Rightarrow \quad 1 &> \text{M}(z, w, t) \\
&\geq \text{M}(z, z, \tfrac{t}{2}) * \text{M}(z, w, \tfrac{t}{2}) \\
&\geq 1 * \text{M}(z, z, \tfrac{t}{4}) * \text{M}(z, w, \tfrac{t}{4}) \\
&\geq 1 * 1 * \text{M}(z, z, \tfrac{t}{8}) * \text{M}(z, w, \tfrac{t}{8}) \\
&\geq 1 * 1 * 1 * \text{M}(z, z, \tfrac{t}{2^4}) * \text{M}(z, w, \tfrac{t}{2^4}) \\
&\ . \\
&\ . \\
&\ . \\
&\geq 1 * 1 * 1 * \dots * 1 * \text{M}(z, w, \tfrac{t}{2^k}) \\
&\to 1 * 1 * 1 * \dots * 1 * 1 \text{ as } k \to \infty \qquad \text{(since from (2.1.2))} \\
&= 1
\end{aligned}$$

$\Rightarrow \quad z = w \Rightarrow z$ is the unique fixed point of T.

Example 2.2: Let X = [0, 2], $\text{M}(x, y, t) = e^{\frac{-d(x,y)}{t}} (0 < t < 1)$ where $\text{d}(x, y) = |x - y|$ and $*$ be the continuous t –norm defined by $a * b = \min\{a, b\}$.

Clearly, (X, M, $*$) is a complete fuzzy metric space.

Let T be a self map on X given by T$x = 2 - x$.

Here, (2.1.1) and (2.1.2) of theorem 2.1 are satisfied by above defined M(x, y, t) and T.

Thus, by theorem 2.1, we can see that T has a unique fixed point in X i.e., at $x = 1$.

IV. CONCLUSION

In this article, we have proved the existence of a fixed point in a fuzzy metric space and showed that the fixed point obtained will be unique if it further satisfies an additional condition.

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