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New Methods for Finding the nth Root of a Number

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ABSTRACT: New methods for finding the n^{th} root of a positive number *m*, to any degree of accuracy, are discussed. These methods are based on finding eigen values and eigen vectors of a special matrix. For even order matrices, the method is founded on the well-known power method. The desired root and its higher powers can also be obtained from the same matrix.

KEYWORDS: Iterative algorithm, Diagonalization, Power method, Dominant Eigen Value.

AMS Classification: 65D99

I. INTRODUCTION

The n^{th} root of a positive number *m* is a number *p* satisfying $p^n = m$. Any real number *m* has *n* such nth roots. In this paper, we are concerned with the numerical approximation of $\sqrt[n]{m}$. There are numerical methods, such as, the bisection method, the regula-falsi method, the Newton-Raphson's method and many more. Any standard textbook on numerical analysis explains these methods [1]. All these methods are applied to the function $f(x) = x^n - m$.

In [2], an iterative algorithm for finding the $\sqrt{m}$ is discussed, which involves generating a sequence of approximations to $\sqrt{m}$. The method is also directly related to the continued fraction representation of $\sqrt{m}$. The convergence of this method is established by studying the eigen values and eigen vectors of a matrix, directly related to the algorithm itself. The approximations are then obtained from the following sequence of fractions:

$$\frac{a}{b} \to \frac{a+mb}{a+b}, \qquad (1)$$

which can also be viewed as a sequence generated from

$$\gamma \to \frac{\gamma+m}{\gamma+1}, \qquad (2)$$

where $\gamma = \frac{a}{b}$. If we consider any fraction as a two dimensional vector, then $\frac{a}{b}$ represents $\begin{bmatrix} a \\ b \end{bmatrix}$ and vice-versa. The right hand expression in relation (1) is then equivalent to the matrix product

$$\begin{bmatrix} 1 & m \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \qquad (3)$$

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Therefore, the successive generation of the sequence of approximations to $\sqrt{m}$, involve the multiplication of higher powers of the square matrix in (3). The convergence of the iterative algorithm directly depends on the nature of the eigen values and eigen vectors of the matrix.

Based on linear algebra concepts, [3] generealizes and mathematically proves the matrix method of [2]. This generalized form of the square matrix in (3) is given to be

$$A_n = \begin{bmatrix} 1 & m & m & \cdots & m & m & m \\ 1 & 1 & m & \cdots & m & m & m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & \cdots & 1 & m & m \\ 1 & 1 & \cdots & \cdots & 1 & 1 & m \end{bmatrix} \qquad (4)$$

Further, using the auto-correlation and the cross-correlation ideas, $\sqrt[n]{m^u}$ is obtained quickly [3].

This paper explores the possibility of finding $\sqrt[n]{m^u}$ by using inverse of the matrix (4). This method is explained in section II. Several examples are included for clarity purposes.

II. COMPUTING THE n^{th} ROOT OF M USING MATRICES

We compute the n^{th} root of a number m, using the inverse matrix $M = A_n^{-1}$. The procedure for finding M is follows.

In [3], the diagonalization of the matrix A_n is obtained in the form as $A_n = S\Lambda S^{-1}$, where S is the eigen vector matrix and Λ is the eigen vaue matrix. These matrices have closed form expression and are given to be:

$$\Lambda_n = \begin{bmatrix} \sum_{j=0}^{n-1}(\omega^0 a)^j & 0 & 0 & \cdots & 0 \\ 0 & \sum_{j=0}^{n-1}(\omega a)^j & 0 & \cdots & 0 \\ 0 & 0 & \sum_{j=0}^{n-1}(\omega^2 a)^j & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{j=0}^{n-1}(\omega^{n-1} a)^j \end{bmatrix},$$

$$S_n = \begin{bmatrix} (\omega^0 a)^{n-1} & (\omega a)^{n-1} & (\omega^2 a)^{n-1} & \cdots & (\omega^{n-1} a)^{n-1} \\ (\omega^0 a)^{n-2} & (\omega a)^{n-2} & (\omega^2 a)^{n-2} & \cdots & (\omega^{n-1} a)^{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (\omega^0 a) & (\omega a) & (\omega^2 a) & \cdots & (\omega^{n-1} a) \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

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Also

$$S_n^{-1} = \frac{1}{nm}\begin{bmatrix} (\omega^0 a) & (\omega^0 a)^2 & (\omega^0 a)^3 & \cdots & (\omega^0 a)^n \\ (\omega a) & (\omega a)^2 & (\omega a)^3 & \cdots & (\omega a)^n \\ (\omega^2 a) & (\omega^2 a)^2 & (\omega^2 a)^3 & \cdots & (\omega^2 a)^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (\omega^{n-1} a) & (\omega^{n-1} a)^2 & (\omega^{n-1} a)^3 & \cdots & (\omega^{n-1} a)^n \end{bmatrix}.$$

The entries of the eigen value matrix and those of the eigen vector matrix are derived to be

$$s_{i,j} = (\omega^{j-1}a)^{n-i}$$

$$s_{i,j}^{-1} = \frac{(\omega^{i-1}a)^j}{nm}$$

And $\lambda_{k,k} = \sum_{j=0}^{n-1}(\omega^{k-1}a)^k = \frac{m-1}{\omega^{i-1}a-1}$.

Thus in order to obtain A_n^{-1}, we find $A_n^{-1} = S^{-1}\Lambda^{-1}S$. The inversion of the eigen value matrix is straight forward. The inverse of the eigen vector matrix A_n^{-1} is computed as explained below. It is verified that the $(i,j)^{th}$ entry of A_n^{-1} is

$$a_{i,j}^{-1} = \sum_{k=1}^{n}(\omega^{k-1}a)^{n-1}\lambda_{k,k}{}^{-1}\frac{(\omega^{k-1}a)^j}{nm} = \sum_{k=1}^{n}(\omega^{k-1}a)^{j-i}\frac{(\omega^{k-1}a-1)}{n(m-1)}$$

$$= \frac{1}{n(m-1)}\sum_{k=1}^{n}(\omega^{k-1}a)^{j-i+1} - (\omega^{k-1}a)^{j-i}$$

For

$$j = i, \qquad a_{i,j}^{-1} = -\frac{1}{m-1}$$

$$and\ j = i+1, \qquad a_{i,j}^{-1} = \frac{1}{m-1}$$

$$and\ j-i+1 = n, \qquad a_{i,j}^{-1} = \frac{m}{m-1}$$

Finally, M is computed to be

$$M = \begin{bmatrix} -\frac{1}{m-1} & 0 & 0 & \cdots & 0 & \frac{m}{m-1} \\ \frac{1}{m-1} & -\frac{1}{m-1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{m-1} & -\frac{1}{m-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{m-1} & -\frac{1}{m-1} \end{bmatrix}_{n\times n} \qquad (18)$$

By direct multiplication of expressions (4) and (18), it can be verified that $M = A_n^{-1}$.

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The characteristic polynomial of this matrix is derived to be

$$(m-1)^{n-1}\lambda^{n}+n(m-1)^{n-2}\lambda^{n-1}+\frac{n(n-1)}{2}(m-1)^{n-3}\lambda^{n-2}+\cdots+n\lambda-1=0 \quad (19)$$

It has been verified that by using the formula $|(m-1)\lambda+1|$, where λ is a **real** root of equation (19), we obtain an approximation to $\sqrt[n]{m}$.

Now, if n is even, using the well-known power method for finding the dominant eigen value, we can obtain an approximate value of $\sqrt[n]{m}$, given by the same formula $|(m-1)\lambda+1|$. If n is odd, then the dominant eigen value is not real.

Interestingly, by finding the eigen vectors of the above matrix and then taking the absolute value of the ratio of any two consecutive entries of any eigen vector (including complex numbers), we can obtain an approximate value of $\sqrt[n]{m}$. For instance, if we choose $m=3$ and $n=4$, then one of the eigenvector is computed to be $[-2.27951i, -1.73205, 1.31607i, 1]^T$, where $i^2=-1$. The absolute value of the ratio of the first two numbers is $\left|\frac{-2.2795i}{-1.73205}\right|=1.31607\approx\sqrt[4]{3}$. If we compute $\sqrt[4]{3}$ using the numerically dominant eigenvalue of the matrix in (18), then we obtain $\sqrt[4]{3}=|2(-1.158)+1|=1.316$, where $\lambda=-1.158$.

The algorithm discussed in [4] is quite fast, having an order of convergence of more than two. We can also use this algorithm to find an eigen vector of the matrix M and then consider the ratio of any two consecutive entries of the eigen vector. This method also approximates $\sqrt[n]{m^u}$ and it is also noted that there is no difference in the orders of convergence.

III. CONCLUSION

Several new iterative methods for finding the n^{th} root of a positive number m have been discussed. It has been explained that these new methods depend upon finding eigen values and eigen vectors of some special matrices. It has also been mentioned that for even order matrices, the methods are founded on the well-known power method. Also, the desired root and its higher powers can be obtained from the same matrices.

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