The Edge Version of Geometric Arithmetic Index of Arbitrary Subdivisions of a Wheel

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Research Article

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ABSTRACT

Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariant. In QSAR/QSPR study, physicochemical properties, and topological indices such as Randic, atom-bond connectivity (ABC) and edge version of geometric-arithmetic (GA) index are used to predict the bioactivity of different chemical compounds. A considerable use of graph theory has been used in this area of research. In this paper, we study the degree based molecular topological index the edge version of GA for certain subdivision of wheel graphs. We derive the analytical closed formulae for these classes of structures.

INTRODUCTION

A single number that can be used to characterize some property of the graph of a molecule is called a topological index for that graph. There are numerous topological descriptors that have found some applications in theoretical chemistry, especially in QSPR/QSAR research [1]. The oldest topological index which introduced by Harold Wiener in 1947 is ordinary (vertex) version of Wiener index [2] which is the sum of all distances between vertices of a graph. Also, the edge version of Wiener index which was based on the distance between edges introduced by Iranmanesh et al. in 2008 [3]. One of the most important topological indices is the well-known branching index introduced by Randic [4] which is defined as the sum of certain bond contributions calculated from the vertex degree of the hydrogen-suppressed molecular graphs. Motivated by the definition of Randic connectivity index based on the end-vertex degrees of edges in a connected graph G with the vertex set V (G) and the edge set E (G) [5,6]. Vukicevic and Furtula [7] proposed a topological index named the geometric-arithmetic index (simply GA) as

$$GA(G) = \sum_{\mu v \in E(G)} \frac{\sqrt[2]{d_G(\mu).d_G(v)}}{d_G(\mu) + d_G(v)} \tag{1}$$

where $d_G(\mu)$ denotes the degree of the vertex μ in G. The reader can find more information about the geometric arithmetic index in [7-12]. In the study carried out by Mahmiani et al. [13], the edge version of the geometric-arithmetic index was introduced based on the end-vertex degrees of edges in a line graph of G which is a graph such that each vertex of L(G) represents an edge of G; and two vertices of L(G) are adjacent if and only if their corresponding edges share a common endpoint in G, as follows

$$GA_e(G) = \sum_{\mu v \in E(L(G))} \frac{\sqrt[2]{d_{L(G)}(\mu).d_{L(G)}(v)}}{d_{L(G)}(\mu) + d_{L(G)}(v)} \tag{2}$$

where $d_{L(G)}(\mu)$ denotes the degrees of an edge μ of the line graph of G.

The edge version of GA index of the benzenoid graph was studied by Farahaini [14]. The total version of GA index was considered in [15,16]. Carbon nanotubes form an interesting class of carbon nanomaterials. There are three types of nanotubes: armchair, chiral, and zigzag structures. Carbon nanotubes show remarkable mechanical properties. Experimental studies have

shown that they belong to the stiffest and elastic known materials. Diudea was the first chemist who considered the problem of computing topological indices of nanostructures [17-20].

A graph can be recognized by a numeric number, a polynomial, a drawing, a sequence of numbers or a matrix. A topological index is a numeric quantity associated with a graph which characterizes the topology of the graph and is invariant under graph automorphism. There are some major classes of topological indices such as distance based topological indices, degree based topological indices and counting related polynomials and indices of graphs. Among these classes degree based topological indices are of great importance and play a vital role in chemical graph theory and particularly in chemistry. The more precise way, a topological index Top (G) of a graph, is a number with the property that for every graph H isomorphic to G, we have Top (H) = Top (G). The concept of topological indices came from the work done by Wiener while he was working on boiling point of paraffin, named this index as path number. Later, the path number was renamed as Wiener index and the whole theory of topological indices started. In this paper, we have investigated the new version of geometric arithmetic index of arbitrary subdivisions of a molecular wheel graph of order n.

Definition 1. Let e be an edge with end points $\{\mu,v\}$ of a graph G. Subdividing the edge e means that a new vertex w is added to $V(G)$ and the edge e is replaced in $E(G)$ by an edge e' with end points $\{\mu,w\}$ and an edge e'' with end points $\{w,v\}$.

Definition 2. Subdividing a graph G means performing a sequence of edge-subdivision operations. The resulting graph is called a subdivision of the graph G. A subdivision of a graph is a graph formed by subdividing its edges into paths of one or more edges.

Example 1. Performing any k-subdivisions on the n-cycle graph C_n yields the $(n+k)$-cycle graph C_{n+K}.

Definition 3. The barycentric subdivision of a graph G is the subdivision in which one vertex is inserted in the interior of each edge.

MAIN RESULTS AND DISCUSSION

In this section, we first give the edge version of GA index of some standard graphs and then the main results of this paper will be given.

Example 2. Let P_n be a path with n vertices. Then, the edge version of the geometric arithmetic index of P_n is GAe(P_n)=

$$\frac{4\sqrt{2}}{3} + (n-4)$$

Example 3. Let S_n be a star graph with n vertices. Then, the edge version of the geometric arithmetic index of S_n is

$$GA_e(S_n) = \frac{(n-1)}{2}$$

Example 4. Let K_n be a complete graph with n vertices. Then, the edge version of the geometric arithmetic index of complete graph, K_n is $GA_e(K_n) = \frac{n(n-1)^2}{2}$

Example 5. Let C_n be a cycle with n vertices. Then, the edge version of geometric arithmetic index of C_n is GAe(K_n)=n.

Example 6. Let W_n be the wheel graph with n vertices. Then, the edge version of geometric arithmetic index of W_n is

$$GA_e(W_n) = (n-1)(1 + \frac{8\sqrt{n}}{n+4} + \frac{n-2}{2})$$

Subdivision of the Cycle of the Wheel Graph W_n

In this subsection, we will give the edge version of the subdivision of a wheel graph with respect to the subdivision on the cyclic edges only. Firstly, we consider the following example.

Example 7. Let $W_{5,1}$ be the graph obtained by the barycentric subdivision of the cycle of the wheel of order 5 as depicted in **Figure 1**. It is easy to see that the line graph of $W_{5,1}$ has 22 edges. On the hand, there are 8 edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=3$, 8 edges of type $d_{L(G)}(\mu)=3$, $d_{L(G)}(v)=5$, and 6 edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=5$.

The edge version of geometric arithmetic index of $W_{5,1}$ is given as: $GA_e(W_{5,1}) = 4(\frac{\sqrt{15}}{2} + \frac{7}{2})$
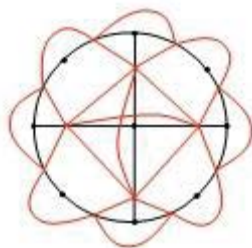
Figure 1. $W_{5,1}$ and its line graph is $L(W_{5,1})$ in red color.

Lemma 1. Let W_n be the wheel graph with n vertices and $W_{n,1}$ is the graph obtained after the barycentric subdivision of the cycle of the wheel. Then

$$GA_e(W_{n,1}) = (n-1)\left(\frac{4\sqrt{3n}}{n+3} + \frac{n+2}{2}\right)$$

Proof. It is easy to see that the line graph of $W_{n,1}$ has $4(n-1) + |K_{n-1}|$ edges. On the hand, there are $2(n-1)$ edges with $d_{L(G)}(\mu) = d_{L(G)}(v) = 3$, $2(n-1)$ edges of type $d_{L(G)}(\mu) = 3$, $d_{L(G)}(v) = n$, and $|K_{n-1}|$ edges with $d_{L(G)}(\mu) = d_{L(G)}(v) = n$.

Since $GA_e(G) = \sum_{\mu v \in E(L(G))} \frac{\sqrt[2]{d_{L(G)}(\mu) . d_{L(G)}(v)}}{d_{L(G)}(\mu) + d_{L(G)}(v)}$, this implies

$$GA_e(W_{n,1}) = 2(n-1)\left(\frac{2\sqrt{3.3}}{3+3}\right) + 2(n-1)\left(\frac{2\sqrt{3.n}}{3+n}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{n.n}}{n+n}\right)$$

After an easy simplification, we obtain $GA_e(W_{n,1}) = (n-1)\left(\frac{4\sqrt{3n}}{n+3} + \frac{n+2}{2}\right)$

Lemma 2. Let W_n be the wheel graph with n vertices and $W_{n,2}$ is the graph obtained after the barycentric subdivision of the cycle of the wheel. Then, the edge version of geometric arithmetic index of $W_{n,2}$ is given as: $GA_e(G) = \sum_{\mu v \in E(L(G))} \frac{\sqrt[2]{d_{L(G)}(\mu) . d_{L(G)}(v)}}{d_{L(G)}(\mu) + d_{L(G)}(v)}$

Proof. The line graph of $W_{n,2}$ has $5(n-1) + |K_{n-1}|$ edges. On the hand, there are $2(n-1)$ edges of type $dL(G)(\mu) = 2$, $dL(G)(v) = 3$, $(n-1)$ edges with $d_{L(G)}(\mu) = d_{L(G)}(v) = 3$, $2(n-1)$ edges of type $d_{L(G)}(\mu) = 3$, $d_{L(G)}(v) = n$, and $|K_{n-1}|$ edges with $d_{L(G)}(\mu) = d_{L(G)}(v) = n$.

Since $GA_e(G) = \sum_{\mu v \in E(L(G))} \frac{\sqrt[2]{d_{L(G)}(\mu) . d_{L(G)}(v)}}{d_{L(G)}(\mu) + d_{L(G)}(v)}$, this implies

$$GA_e(W_{n,1}) = 2(n-1)\left(\frac{2\sqrt{2.3}}{2+3}\right) + (n-1)\left(\frac{2\sqrt{3.3}}{3+3}\right) + 2(n-1)\left(\frac{2\sqrt{3.n}}{3+n}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{n.n}}{n+n}\right)$$

After an easy simplification, we obtain $GA_e(W_{n,m}) = (n-1)\left((m-2) + \frac{4\sqrt{3n}}{n+3} + \frac{n}{2} + \frac{4\sqrt{6}}{5}\right)$

Example 8. Let $W_{5,3}$ be the graph obtained by the uniform 3-subdivision of the cycle of the wheel of order 5. Then

$$GA_e(W_{n,m}) = (n-1)\left((m-2) + \frac{4\sqrt{3n}}{n+3} + \frac{n}{2} + \frac{4\sqrt{6}}{5}\right)$$

Lemma 3. Let W_n be the wheel graph with n vertices and $W_{n,m}$ is obtained after the uniform m-subdivision of the cycle of the wheel by m vertices. Then, the edge version of geometric arithmetic index of $W_{n,m}$ is given as: $GA_e(W_{n,m}) = (n-1)\left((m-2) + \frac{4\sqrt{3n}}{n+3} + \frac{n}{2} + \frac{4\sqrt{6}}{5}\right)$

Proof. The line graph of $W_{n,2}$ has $(m+3)(n-1) + |K_{n-1}|$ edges. On the hand, there are $(m-2)(n-1)$ edges of type $d_{L(G)}(\mu) = d_{L(G)}(v) = 2$, $2(n-1)$ edges of type $d_{L(G)}(\mu) = 2$, $d_{L(G)}(v) = 3$, $(n-1)$ edges with $d_{L(G)}(\mu) = d_{L(G)}(v) = 3$, $2(n-1)$ edges of type $d_{L(G)}(\mu) = 3$, $d_{L(G)}(v) = n$, and $|K_{n-1}|$ edges with $d_{L(G)}(\mu) = d_{L(G)}(v) = n$.

Since, $GA_e(G) = \sum_{\mu v \in E(L(G))} \frac{\sqrt[2]{d_{L(G)}(\mu) . d_{L(G)}(v)}}{d_{L(G)}(\mu) + d_{L(G)}(v)}$, this implies

$$GA_e(W_{n,m}) = (m-2)(n-1)\left(\frac{2\sqrt{2.2}}{2+2}\right) + 2(n-1)\left(\frac{2\sqrt{2.3}}{2+3}\right) + (n-1)\left(\frac{2\sqrt{3.3}}{3+3}\right) + 2(n-1)\left(\frac{2\sqrt{3.n}}{3+n}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{n.n}}{n+n}\right)$$

After an easy simplification, we obtain $GA_e(W_{n,m}) = (n-1)\left((m-2) + \frac{4\sqrt{3n}}{n+3} + \frac{n}{2} + \frac{4\sqrt{6}}{5}\right)$

Example 9. Let $W_{5,2,3,4,5}$ be the graph obtained by the subdivision of the cycle of the wheel of order 5. Then $GA_e(G) = 14 + 4\left[\frac{\sqrt{15}}{2} + \frac{1}{2} + \frac{4\sqrt{6}}{5}\right]$

Theorem 1. Let W_n be the wheel graph with n vertices and the graph G=$W_{n,k1,k2,k3,...,kn-1}$ is obtained by sub diving the edges of cycle of the wheel by $k_1, k_2, k_3, ..., k_{n-1}$ vertices, where $k_i \geq 3$ for each $i=1,2,3,...,n-1$. Then, the edge version of geometric arithmetic index of the graph G is given as:

$$GA_e(G) = \sum_{i=1}^{n-1} k_i + (n-1)\left[\frac{ü\ddot{w}\sqrt{n}}{n+3} + \frac{n-}{2} + \frac{\sqrt{}}{5}\right]$$

Proof. Since the line graph of a path P_{ki} of length k_i is again a path of length k_{i-1}, so the line graph of $W_{n, k1,k2,k3,...,kn-1}$ has

$\sum_{i=1}^{n-1}(k_i - 2) + 5(n-1) + |K_{n-1}|$ edges. On the hand, there are $\sum_{i=1}^{n-1}(k_i - 2)$ edges of type $d_{L(G)}(\mu) = d_{L(G)}(v) = 2$, $2(n-1)$ edges of type $d_{L(G)}(\mu) = 2$, $d_{L(G)}(v) = 3$, $(n-1)$ edges with $d_{L(G)}(\mu) = d_{L(G)}(v) = 3$, $2(n-1)$ edges of type $d_{L(G)}(\mu) = 3$, $d_{L(G)}(v) = n$, and $|K_{n-1}|$ edges with $d_{L(G)}(\mu) = d_{L(G)}(v) = n$.

Since, $GA_e(G) = \sum_{\mu v \in E(L(G))} \frac{2\sqrt{d_{L(G)}(\mu).d_{L(G)}(v)}}{d_{L(G)}(\mu) + d_{L(G)}(v)}$, this implies

$$GA_e(W_{n,k_1,k_2,k_3,...,k_{n-1}}) = \sum_{i=1}^{n-1}(k_i - 2)\left(\frac{2\sqrt{2.2}}{2+2}\right) + 2(n-1)\left(\frac{2\sqrt{2.3}}{2+3}\right) + (n-1)\left(\frac{2\sqrt{3.3}}{3+3}\right) + 2(n-1)\left(\frac{2\sqrt{3.n}}{3+n}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{n.n}}{n+n}\right)$$

After an easy simplification, we obtain $GA_e(G) = \sum_{i=1}^{n-1} k_i + (n-1)\left[\frac{4\sqrt{3n}}{n+3} + \frac{n-4}{2} + \frac{4\sqrt{6}}{5}\right]$.
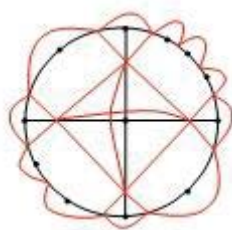
Let G=$W_{n, k1,k2,k3,...,kn-1}$ be a graph obtained from the subdivision of the edges of the cycle of a wheel graph W_n of order n, where, $k_i=1$, for each $i=1,2,3,...,t$ and $k_i \geq 2$ for each $i=t+1,t+2,...,n-1$.

In fact, we can relabelled the number of edges on the cycle of the wheel so that, so that we get the desired label of the subdivision. The edge partitions of the graph G are as follows:

$E4 = \{\mu v \in E(L(G)): d_{L(G)}(\mu) = d_{L(G)}(v) = 2\}$, so we have $|E4| = \sum_{i=t+1}^{n-1}(k_i - 2)$
$E6 = \{\mu v \in E(L(G)): d_{L(G)}(\mu) = 2 \& d_{L(G)}(v) = 3\}$, so we have $|E6| = 2(n-1) - 2t$,
$E9 = \{\mu v \in E(L(G)): d_{L(G)}(\mu) = 3 \& d_{L(G)}(v) = 3\}$, so we have $|E9| = (n-1) + t$,
$E_{3n} = \{\mu v \in E(L(G)): d_{L(G)}(\mu) = 3 \& d_{L(G)}(v) = n\}$, so we have $|E_{3n}| = 2(n-1)$,
$E_{n.n} = \{\mu v \in E(L(G)): d_{L(G)}(\mu) = n \& d_{L(G)}(v) = n\}$, so we have $|E_{n.n}| = |K_{n-1}|$.

Example 10. Let $W_{5,1,1,2,3}$ be the graph obtained by the subdivision of the cycle of the wheel of order 5 as depicted in **Figure 2**. Then, the edge version of geometric arithmetic index of $W_{5,1,1,2,3}$ is given as:

$$GA_e(W_{5,1,1,2,3}) = 1 + \left(\frac{5-4\sqrt{6}}{5}\right)2 + 4\left(1 + \frac{\sqrt{15}}{2} + \frac{3}{2} + \frac{4\sqrt{6}}{5}\right)$$

Figure 2. $W_{5,1,1,2,3}$ and its line graph is $L(W_{5,1,1,2,3})$ in red color.

Theorem 2. Let $W_{n,k1,k2,k3,...,kn-1}$ be a graph obtained from the subdivision of a wheel graph of order n, where, $k_i=1$, for each $i=1,2,3,...,t$ and $k_i \geq 2$ for each $i=t+1,t+2,...,n-1$. Then, the edge version of geometric arithmetic index of $W_{n,k1,k2,k3,...,kn-1}$ is given as:

$$GA_e(W_{n,k_1,k_2,k_3,...,k_{n-1}}) = \sum_{i=t+1}^{n-1}(k_i - 2) + \left(\frac{5-4\sqrt{6}}{5}\right)t + (n-1)\left(1 + \frac{4\sqrt{3n}}{n+3} + \frac{n-2}{2} + \frac{4\sqrt{6}}{5}\right)$$

Proof. From Equation 2, we have

$$GA_e(W_{n,k_1,k_2,k_3,...,k_{n-1}}) = |E_4|\left(\frac{2\sqrt{2.2}}{2+2}\right) + |E_6|\left(\frac{2\sqrt{2.3}}{2+3}\right) + |E_9|\left(\frac{2\sqrt{3.3}}{3+3}\right) + |E_{3n}|\left(\frac{2\sqrt{3.n}}{3+n}\right) + |E_{n.n}|\left(\frac{2\sqrt{n.n}}{n+n}\right)$$

Using the cardinalities of the edge partitions and after simplifications, we get

$$GA_e(G) = \sum_{i=t+1}^{n-1}(k_i - 2) + \left(\frac{5-4\sqrt{6}}{5}\right)t + (n-1)\left(1 + \frac{4\sqrt{3n}}{n+3} + \frac{n-2}{2} + \frac{4\sqrt{6}}{5}\right)$$

Subdivision of the Spoke of the Wheel Graph W$_n$

In this subsection, we will give the edge version of the subdivision of a wheel graph with respect to the subdivision on the spoke of the wheel only.

Example 11. Let $\left(W_5^1\right)$ be the graph obtained by the barycentric subdivision of the spoke of the wheel of order 5. Then

$$GA_e\left(W_5^1\right) = 4\left(\frac{6\sqrt{12}}{7} + \frac{5}{2}\right)$$

Lemma 4. Let W_n be the wheel graph with n vertices and $\left(W_n^2\right)$ is the barycentric subdivision of the spokes of the wheel. Then

$$GA_e(W_n^1) = (n-1)\left(\frac{4\sqrt{12}}{7} + \frac{2\sqrt{3n-3}}{n+2} + \cdot\right)$$

Proof. By continuing an induction argument on n, one can check that in general, the line graph of $\left(W_n^1\right)$ has 4(n-1) + $|K_{n-1}|$ edges. On the hand, there are $(n-1)$ edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=4$, 2(n-1) edges of type $d_{L(G)}(\mu)=3$, $d_{L(G)}(v)=4$, (n-1) edges of type $d_{L(G)}(\mu)=3$, $d_{L(G)}(v)=n-1$ and $|K_{n-1}|$ edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=n-1$.

So, by Equation 2, we have

$$GA_e(W_n^1) = (n-1)\left(\frac{2\sqrt{4.4}}{4+4}\right) + 2(n-1)\left(\frac{2\sqrt{3.4}}{3+4}\right) + (n-1)\left(\frac{2\sqrt{3n-3}}{3+n-1}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{(n-1)(n-2)}}{n-1+n-1}\right)$$

After an easy simplification, we obtain $GA_e(W_n^1) = (n-1)\left(\frac{4\sqrt{12}}{7} + \frac{2\sqrt{3n-3}}{n+2} + \frac{n}{2}\right)$.

Lemma 5. Let W_n be the wheel graph with n vertices $\left(W_n^2\right)$ is the uniform 2-subdivision of the spokes of the wheel. Then, the edge version of geometric arithmetic index of $\left(W_n^2\right)$ is given by:

$$GA_e(W_n^2) = (n-1)\left(\frac{4\sqrt{12}}{7} + \frac{2\sqrt{6}}{5} + \frac{2\sqrt{2n-2}}{n+1} + \frac{n}{2}\right).$$

Proof. By continuing an induction argument on n, one can check that in general, the line graph of $\left(W_n^2\right)$ has 5(n - 1) + $|K_{n-1}|$ edges. On the hand, there are $(n-1)$ edges of type $d_{L(G)}(\mu)=d_{L(G)}(v)=4$, 2(n-1) edges with $d_{L(G)}(\mu)=3$ & $d_{L(G)}(v)=4$, (n-1) edges of type $d_{L(G)}(\mu)=2$ & $d_{L(G)}(v)=3$, (n-1) edges of type $d_{L(G)}(\mu)=2$, & $d_{L(G)}(v)=n-1$ and $|K_{n-1}|$ edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=n-1$.

Since, $GA_e(G) = \sum_{\mu v \in E(L(G))} \frac{\sqrt[2]{d_{L(G)}(\mu).d_{L(G)}(v)}}{d_{L(G)}(\mu) + d_{L(G)}(v)}$, this implies

$$GA_e(W_n^2) = (n-1)\left(\frac{2\sqrt{4.4}}{4+4}\right) + 2(n-1)\left(\frac{2\sqrt{3.4}}{3+4}\right) + (n-1)\left(\frac{2\sqrt{2.3}}{2+3}\right) + (n-1)\left(\frac{2\sqrt{2n-2}}{2+n-1}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)}\right)$$

After an easy simplification, we obtain $GA_e(W_n^2) = (n-1)\left(\frac{4\sqrt{12}}{7} + \frac{2\sqrt{6}}{5} + \frac{2\sqrt{2n-2}}{n+1} + \frac{n}{2}\right).$

Theorem 3. Let W_n be the wheel graph with n vertices W_n^m is the uniform m-subdivision of the spokes of the wheel by m>2 vertices. Then, the edge version of geometric arithmetic index of W_n^m is given as: $GA_e(W_n^m) = (n-1)\left((m-1)\frac{2\sqrt{6}}{5} + \frac{4\sqrt{12}}{7} + \frac{4\sqrt{2n-2}}{n+1} + \frac{n-2}{2}\right)$

Proof. It is easy to see that the line graph of Wnm has $(m+3)(n-1)+|K_{n-1}|$ edges. On the hand, there are $(m-2)(n-1)$ edges of type $d_{L(G)}(\mu)=d_{L(G)}(v)=2$, (n-1) edges of type $d_{L(G)}(\mu)=2$, $d_{L(G)}(v)=3$, (n-1) edges of type $d_{L(G)}(\mu)=3$, $d_{L(G)}(v)=4$, (n-1) edges of type $d_{L(G)}(\mu)= d_{L(G)}(v)=4$, 2(n-1) edges of type $d_{L(G)}(\mu)=2$, $d_{L(G)}(v)=n-1$, and $|K_{n-1}|$ edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=n-1$.

Since, $GA_e(G) = \sum_{\mu v \in E(L(G))} \frac{\sqrt[2]{d_{L(G)}(\mu).d_{L(G)}(v)}}{d_{L(G)}(\mu) + d_{L(G)}(v)}$, this implies

$$GA_e(W_n^m) = (m-2)(n-1)\left(\frac{2\sqrt{2.2}}{2+2}\right) + (n-1)\left(\frac{2\sqrt{2.3}}{2+3}\right) + 2(n-1)\left(\frac{2\sqrt{3.4}}{3+4}\right) + (n-1)\left(\frac{2\sqrt{4.4}}{4+4}\right) + (n-1)\left(\frac{2\sqrt{2n-2}}{2+n-1}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)}\right)$$

After simplification, we obtain $GA_e(W_n^m) = (n-1)\left((m-1)\frac{2\sqrt{6}}{5} + \frac{4\sqrt{12}}{7} + \frac{4\sqrt{2n-2}}{n+1} + \frac{n-2}{2}\right).$

Arbitrary Subdivision of Wheel Graph

In this subsection, we will give the edge version of the arbitrary subdivision of a wheel graph.

Lemma 6. Let W_n be the wheel graph with n vertices and $W_{n,1}^1$ is the barycentric subdivision of the wheel graph Then

$$GA_e(W_{n,1}^1) = (n-1)\left((\frac{2\sqrt{3n-3}}{n+2} + \frac{n+6}{2}\right).$$

Proof. By continuing an induction argument on n, one can check that in general, the line graph of $W_{n,1}^1$ has $5(n-1)+|K_{n-1}|$ edges. On the hand, there are $4(n-1)$ edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=3$, $(n-1)$ edges of type $d_{L(G)}(\mu)=3$, $d_{L(G)}(v)=n\text{-}1$ and $|K_{n-1}|$ edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=n\text{-}1$.

Since, $GA_e(G) = \sum\limits_{\mu v \in E(L(G))} \dfrac{\sqrt[2]{d_{L(G)}(\mu).d_{L(G)}(v)}}{d_{L(G)}(\mu)+d_{L(G)}(v)}$, this implies

$$GA_e(W_{n,1}^1) = 4(n-1)\left(\frac{2\sqrt{4.4}}{4+4}\right) + (n-1)\left(\frac{2\sqrt{3n-3}}{3+n-1}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)}\right)$$

After an easy simplification, we obtain $GA_e(W_{n,1}^1) = (n-1)\left[\dfrac{2\sqrt{3n-3}}{n+2} + \dfrac{n+6}{2}\right]$.

Lemma 7. Let W_n be the wheel graph with n vertices and $W_{n,2}^2$ is the 2-uniform subdivision of the wheel graph. Then

$$GA_e(G) = \sum\limits_{\mu v \in E(L(G))} \frac{\sqrt[2]{a_{L(G)}(\mu).a_{L(G)}(v)}}{d_{L(G)}(\mu)+d_{L(G)}(v)} .$$

Proof. By continuing an induction argument on n, one can check that in general, the line graph of W_n has $7(n-1)+|K_{n-1}|$ edges. On the hand, there are $3(n-1)$ edges with $d_{L(G)}(\mu)=2,d_{L(G)}(v)=3$, $3(n-1)$ edges of type $d_{L(G)}(\mu)= d_{L(G)}(v)=3$, $(n-1)$ edges of type $d_{L(G)}(\mu)=2$, $d_{L(G)}(v)=n\text{-}1$ and $|K_{n-1}|$ edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=n\text{-}1$.

Since, $GA_e(G) = \sum\limits_{\mu v \in E(L(G))} \dfrac{\sqrt[2]{d_{L(G)}(\mu).d_{L(G)}(v)}}{d_{L(G)}(\mu)+d_{L(G)}(v)}$, this implies

$$GA_e(W_{n,2}^2) = 3(n-1)\left(\frac{2\sqrt{2.3}}{2+3}\right) + 3(n-1)\left(\frac{2\sqrt{3.3}}{3+3}\right) + (n-1)\left(\frac{2\sqrt{2n-2}}{2+n-1}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)}\right)$$

After an easy simplification, we obtain $GA_e(W_{n,m}^m) = (n-1)\left[2m-1+\dfrac{6\sqrt{6}}{5} + \dfrac{2\sqrt{2n-2}}{n+1} + \dfrac{n-2}{2}\right]$

Theorem 4. Let W_n be the wheel graph with n vertices and $W_{n,m}^m$ is the uniform m>2 -subdivision of the wheel. Then, the edge version of geometric arithmetic index of $W_{n,m}^m$ is given as

$$GA_e(W_{n,m}^m) = (n-1)\left[2m-1+\frac{6\sqrt{6}}{5} + \frac{2\sqrt{2n-2}}{n+1} + \frac{n-2}{2}\right]$$

Proof. By continuing an induction argument on n, one can check that in general, the line graph of Wn has $(2m+3)(n-1)+|K_{n-1}|$ edges. On the hand, there are $2(m-2)(n-1)$ edges of type $d_{L(G)}(\mu)=d_{L(G)}(v)=2$, $3(n-1)$ edges of type $d_{L(G)}(\mu)=2$, $d_{L(G)}(v)=3$, $3(n-1)$ edges of type $d_{L(G)}(\mu)= d_{L(G)}(v)=3$, $(n-1)$ edges of type $d_{L(G)}(\mu)=2$, $d_{L(G)}(v)=n\text{-}1$, and $|K_{n-1}|$ edges with $d_{L(G)}(\mu)=d_{L(G)}(v)=n\text{-}1$.

Since, $GA_e(G) = \sum\limits_{\mu v \in E(L(G))} \dfrac{\sqrt[2]{d_{L(G)}(\mu).d_{L(G)}(v)}}{d_{L(G)}(\mu)+d_{L(G)}(v)}$, this implies

$$GA_e(W_{n,m}^m) = 2(m-1)(n-1)\left(\frac{2\sqrt{2.2}}{2+2}\right) + 3(n-1)\left(\frac{2\sqrt{2.3}}{2+3}\right) + 3(n-1)\left(\frac{2\sqrt{3.3}}{3+3}\right) + (n-1)\left(\frac{2\sqrt{2n-1}}{2+n-1}\right) + \frac{(n-1)(n-2)}{2}\left(\frac{2\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)}\right)$$

After an easy simplification, we obtain $GA_e(W_{n,m}^m) = (n-1)\left[2m-1+\dfrac{6\sqrt{6}}{5} + \dfrac{2\sqrt{2n-1}}{n+1} + \dfrac{n-2}{2}\right]$

Let $|E_1| = \sum\limits_{i=t+1}^{n-1}(k_i-2) + \sum\limits_{i=s+1}^{n-1}(l_i-2)$ be a graph obtained from the subdivision of the edges of the wheel graph W_n of order n, where, $k_i=1$, for each $i=1,2,3,...,t$ and $k_i \geq 2$ for each $i=t+1,t+2,...,n\text{-}1$, for the edges of cycle of the wheel and $l_i=1$, for each $i=1,2,3,...,s$ and $l_i \geq 2$ for each $i=s+1,s+2,...,n\text{-}1$ for the spokes of the wheel.

The edge partitions of the graph G are as follows:

$E1= \{\mu v \in E(L(G)): d_{L(G)}(\mu)=d_{L(G)}(v)=2\}$, so we have $|E_1| = \sum\limits_{i=t+1}^{n-1}(k_i-2) + \sum\limits_{i=s+1}^{n-1}(l_i-2)$
$E2= \{\mu v \in E(L(G)): d_{L(G)}(\mu)=2 \ \& \ d_{L(G)}(v)=3\}$, so we have $|E2|=3(n-1)-2t-s$
$E3= \{\mu v \in E(L(G)): d_{L(G)}(\mu)=3 \ \& \ d_{L(G)}(v)=3\}$, so we have $|E3|=3(n-1)+t$
$E4= \{\mu v \in E(L(G)): d_{L(G)}(\mu)=2 \ \& \ d_{L(G)}(v)=n\text{-}1\}$, so we have $|E4|=(n-1)-s$
$E5= \{\mu v \in E(L(G)): d_{L(G)}(\mu)=3 \ \& \ d_{L(G)}(v)=n\text{-}1\}$, so we have $|E5|=s$
$E6= \{\mu v \in E(L(G)): d_{L(G)}(\mu)=d_{L(G)}(v)=n\text{-}1\}$, so we have $|E6|=|K_{n-1}|$.

Example 12. Let $W_{5,1,1,2,3}^{1,2,3,4}$ be the graph obtained by the subdivision of the wheel of order 5 as depicted in **Figure 3**. Then

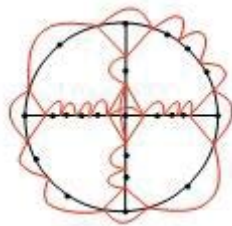$$GA_e(W_{5,1}^1) = 4\left[\frac{4\sqrt{12}}{7} + \frac{2\sqrt{6}}{5} + \frac{\sqrt{8}}{3} + \frac{5}{2}\right]$$

Figure 3. $W_{5,1,1,2,3}^{1,2,3,4}$ and its line graph is $L(W_{5,1,1,2,3}^{1,2,3,4})$ in red color.

Theorem 5. Let $G = W_{n,k_1,k_2,k_3,\dots,k_{n-1}}^{l_1,l_2,l_3,\dots,l_{n-1}}$ be a graph obtained from the subdivision of a wheel graph. Then, the edge version of geometric arithmetic index of G is given as: $GA_e(G) = \sum_{i=t+1}^{n-1}(k_i-2) + \sum_{i=t+1}^{n-1}(k_i-2) + \left(\frac{5-4\sqrt{6}}{5}\right)t + (n-1)\left(1 + \frac{4\sqrt{3n}}{n+3} + \frac{n-2}{2} + \frac{4\sqrt{6}}{5}\right)$

Proof. From Equation 1, we get

$$GA_e(G) = \sum_{i=t+1}^{n-1}(k_i-2) + \sum_{i=s+1}^{n-1}(l_i-2) + \left(\frac{5-4\sqrt{6}}{5}\right)t + 2s\left(\frac{\sqrt{3n-3}}{n+2} - \frac{2\sqrt{2n-2}}{n+1} - \frac{6\sqrt{6}}{5}\right) + (n-1)\left(\frac{6\sqrt{6}}{5} + \frac{2\sqrt{2n-2}}{n+1} + \frac{n+4}{2}\right)$$

Using the cardinalities of the edge partitions and after simplifications, we get

$$GA_e(G) = \sum_{i=t+1}^{n-1}(k_i-2) + \sum_{i=s+1}^{n-1}(l_i-2) + \left(\frac{5-4\sqrt{6}}{5}\right)t + 2s\left(\frac{\sqrt{3n-3}}{n+2} - \frac{2\sqrt{2n-2}}{n+1} - \frac{6\sqrt{6}}{5}\right) + (n-1)\left(\frac{6\sqrt{6}}{5} + \frac{2\sqrt{2n-2}}{n+1} + \frac{n+4}{2}\right)$$

CONCLUSION

In Theoretical Chemistry, the topological indices and molecular structure descriptors are used for modeling physicochemical, toxicological, biological, and other properties of chemical compounds. In recent years, some researchers are interested in studying the topological indices of certain nanotubes and nanotori, for example, see [21,22]. In this paper, we have investigated the new version of the geometric arithmetic index of arbitrary subdivisions of a wheel graph of order n.

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