Volume 5, No. 2, February 2014

Journal of Global Research in Computer Science

ISSN-2229-371X

REVIEW ARTICLE

Available Online at www.jgrcs.info

ROLE OF FRACTIONAL CALCULUS TO THE GENERALIZED INVENTORY MODEL

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Abstract: The feature of the derivative and integration is one of the most important tools to realise the beauty of calculus. Its descriptive power comes from the fact that it analyses the behaviour at scales small enough that its properties changes linearly, so avoiding complexities that arises at larger one. Fractional Calculus generalizes this concept from integer to non integer order. This paper comprises an application of this Fractional Calculus in Inventory Control model. The object of this note is to show how fractional calculus approach can be employed to generalize the traditional classical inventory model. Lastly a numerical example is given.

Keywords: Fractional differentiation, Fractional Integration, Fractional Differential Equation, Set up Cost, Holding Cost, Economic Order Quantity.

INTRODUCTION

Fractional Calculus (FC) is a branch of applied mathematics that deals with generalization of the operation of derivatives/integrals and differential equation of an arbitrary order (including complex orders)[8],[13],[24],[27]. The theory of FC is one of the strongest tool to describe many physical phenomena which are neglected in the model described by the classical integer order calculus. This is almost three centuries old as the conventional calculus but not very popular among science or engineering community. Since then the subject of Fractional Calculus caught the attention of many great mathematicians (pure and applied) such as N.H.Abel, L.Euler, J.Fourier, H.K.Grunwald, J.Hadamarad, G.H.Hardy, O.Heviside, P.S.Laplace, G.W.Lebinitz, A.V.Letnikov , B.Riemann, J.Liouville, M.Caputo, M.Reisz and H.Weyl are directly or indirectly contributed to its development. The mathematics involving fractional order derivatives or integrals are appeared very different from that of integer order calculus. Initially there were almost no practical application of this field and due to this it was considered that fractional calculus as an abstract area containing only rigorous mathematical manipulations. So for past three centuries this subject was with only mathematicians. But in recent years(forty years almost) this subject has been applied to several fields of engineering, science and economics[11]. Some of the areas where Fractional Calculus has made an important role that are included viscoelasticity and rheology [3], electrical engineering[14], electrochemistry, biology, biophysics and bioengineering, electromagnetic theory[15], mechanics, fluid mechanics[12], signal and image processing theory[6], particle physics, control theory[14] and many other field[1],[20],[4],[10],[23]. However there are some areas of management and science where this branch of mathematics remains untouched.

The application of Fractional Calculus and Fractional Differential equation are not being used so far in any Operation Research model. Our objective in this paper is to develop the traditional classical EOQ based inventory model [31[,[32],[9],[2]. to a generalized EOQ based inventory model emphasis on some certain assumption by using the potential application of Fractional Calculus. In traditional EOQ based inventory model[2], the demand(deterministic) rate is typically assumed to be of fixed 1st order differentiation and holding cost is calculated on the fixed 1st order integral of inventory level. We may develop our consideration by accepting that the demand rate may varies high or low according to the market situation. So here we may consider that demand rate may be taken as fractional order differentiation instead of fixed 1st order differentiation. Similarly we calculate the holding cost is of fractional order integral of inventory level rather than that of fixed 1st order integral.

Here we have applied the concept of derivative/integrals with an emphasis on Caputo and Riemann-Liouville fractional derivatives and have some interesting result and ideas that demonstrate the generalized EOQ based inventory model. Fractional derivatives and fractional integrals have interesting mathematical properties that may be utilized to developed our motivation. In this article, first we give a brief historical review of the general principles, definitions and several features of fractional derivatives/integrals and then we review some of our ideas and findings in exploring potential applications of fractional calculus in inventory control model.

BRIEF HISTORY RELATED TO FRACTIONAL CALCULUS

As to history of Fractional Calculus, already in 1695 L'Hospital raised the question to Lebinitz, as the meaning of $\frac{d^n y}{dx^n}$ if n=1/2, that is "what if n is fractional?" Lebinitz replied "This is an apparent paradox from which one day useful consequence will be drawn".

S.F Lacroix was the first to mention in some two pages a derivative of arbitrary order in a 700 pages text book of 1819.

He developed the formula for the nth derivative of y= x^m , m is a positive integer,

$$D^n y = \frac{m!}{(m-n)!}\, x^{m-n}$$, where n(≤m) is an integer. (2.1)

Replacing the factorial symbol by the well known Gamma function, he obtained the formula for the fractional derivative,

$$D^{\alpha}(x^{\beta}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \qquad (2.2)$$

Where α, β are fractional numbers.

In particular he had, $D^{1/2}(x) = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{1/2} = 2\sqrt{\frac{x}{\pi}}$. (2.3)

Again the normal derivative of a function f is defined as,

$$D^1 f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \qquad (2.4)$$

And $D^2 f(x) = \lim_{h\to 0} \frac{f^1(x+h) - f^1(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+2h) - f(x+h) + f(x)}{h}.$$

Iterating this operation yields an expression for the nth derivative of a function. As can be easily seen and proved by induction for any natural number n,

$$D^n f(x) = \lim_{h\to 0} h^{-n} \sum_{r=0}^{n} (-1)^n \binom{n}{r} f(x+(n-r)h). \qquad (2.5)$$

Where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ (2.6)

Or equivalently,

$$D^n f(x) = \lim_{h\to 0} h^{-n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x-rh) \qquad (2.7)$$

The case of n=0 can be included as well.

The fact that for any natural number n the calculation of nth derivative is given by an explicit formula (2.5) or (2.7).

Now the generalization of the factorial symbol (!) by the gamma function allows

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} \qquad (2.8)$$

Which also valid for non integer values.

Thus on using of the idea (2.8), fractional derivative leads as the limit of a sum given by

$$D^{\alpha} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{n} (-1)^r \frac{\Gamma(\alpha+1)}{\Gamma(r+1)\Gamma(\alpha-r+1)} f(x-rh). \qquad (2.9)$$

Provided the limit exists. Using the identity

$$(-1)^r \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)} = \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)} \qquad (2.10)$$

The result (2.9) becomes,

$$D^{\alpha} f(x) = \lim_{h\to 0} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \sum_{r=0}^{n} \frac{\Gamma(r-\alpha)}{\Gamma(r+1)} f(x-rh) \qquad (2.11)$$

When α is an integer, the result (2.9)reduce to the derivative of integral order n as follows in (2.5).

Again in 1927 Marchaud formulated the fractional derivative of arbitrary order α in the form given by,

$$D^{\alpha} f(x) = \frac{f(x)}{\Gamma(1-\alpha)\, x^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x \frac{f(x)-f(t)}{(x-t)^{\alpha+1}} dt$$, Where

0<α<1 (2.12)

In 1987 Samko et al had shown that (2.12) and (2.9) are equivalent.

Replacing n by (-m) in (2.7), it can be shown that

$${}_0D_x^{-m} f(x) = \lim_{h\to 0} h^m \sum_{r=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} f(x-rh)$$

$$= \frac{1}{\Gamma(m)} \int_0^x (x-t)^{(m-1)} f(t)dt \qquad (2.13)$$

Where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{m(m+1)(m+2)\ldots\ldots\ldots(m+r+1)}{r!} \qquad (2.14)$$

This observation naturally leads to the idea of generalization of the notations of differentiation and integration by allowing m in (2.13) to be an arbitrary real or even complex number.

Fractional derivatives and integrals:

The idea of fractional derivative or fractional integral can be described in another different ways.

First, we consider a linear non homogeneous nth order ordinary differential equation ,

$$D^n y = f(x), \quad b \le x \le c \qquad (2.1.1)$$

Then {1, x, x^2, x^3, ...,x^{n-1}} is a fundamental set the corresponding homogeneous equation D^ny=0. f(x) is any continuous function in [b,c], then for any a ∈ (b,c), y(x)= $\int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt$ (2.1.2)

Is the unique solution of the equation (2.1.1) with the initial data $y^{(k)}$(a)=0,

for $0 \le k \le n-1$. Or equivalently, y(x)=

$$ {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt \quad (2.1.3)$$

Replacing n by α ,where Re(α)>0 in the above formula (2.1.3),we obtain the Riemann-Liouville definition of fractional integral that was reported by Liouville in 1832 and by Riemann in 1876 as ${}_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) =$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \quad (2.1.4)$$

Where

$$ {}_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

is the Riemann-Liouville integral operator. When a=0 ,(2.1.4) is the Riemann definition of integral and if a= -∞, (2.1.4) represents Liouville definition. Integral of this type were found to arise in theory of linear ordinary differential equations where they are known as Eulier transform of first kind.

If a=0 and x>0 ,then the Laplace transform solution the initial value problem

$$D^n y(x)=f(x), \quad x>0, \quad y^{(k)}(0)=0, \quad 0 \le k \le n-1 \quad (2.1.5)$$

$$\text{is } \bar{y}(s) = s^{-n} \bar{f}(s) \quad (2.1.6)$$

Where $\bar{y}(s)$ and $\bar{f}(s)$ are respectively the Laplace transform of the function y(x) and f(x).

The inverse Laplace transform gives the solution of the initial value problem (2.1.5) as

$$y(x) = {}_0D_x^{-n} f(x)$$

Again from (2.1.6) we have y(x)= $L^{-1}\{\bar{y}(s)\}$

$= L^{-1}\{s^{-n} \bar{f}(s)\}$

Thus we have ${}_0D_x^{-n} f(x) = L^{-1}\{s^{-n} \bar{f}(s)\}$ (2.1.7)

i.e

$$L^{-1}\{s^{-n} \bar{f}(s)\} = {}_0D_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t)dt \quad (2.1.8)$$

∴ y(x)=

$$ {}_0D_x^{-n} f(x) = L^{-1}\{s^{-n} \bar{f}(s)\} = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t)dt$$

This is the Riemann-Liouville integral formula for an integer n. Replacing n by real α gives the Riemann-Liouville fractional integral (2.1.3) with a=0.

In complex analysis the Cauchy integral formula for the nth derivative of an analytic function f(z) is given by

$$D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt \quad (2.1.9)$$

Where C is closed contour on which f(z) is analytic , and t=z is any point inside C and t=z is a pole.

If n is replaced by an arbitrary number α and n! by $\Gamma(\alpha+1)$, then a derivative of arbitrary order α can be defined by,

$$D^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{f(t)}{(t-z)^{\alpha+1}} dt \quad (2.1.10)$$

where t=z is no longer a pole but a branch point.

In (2.1.10) C is no longer appropriate contour, and it is necessary to make a branch cut along the real axis from the point z=x>0 to negative infinity.

Thus we can define a derivative of arbitrary α order by loop integral

$$ {}_aD_x^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_a^x (t-z)^{-\alpha-1} f(t) \quad (2.1.11)$$

Where $(t-z)^{-\alpha-1}$ = exp[-(α+1)ln(t-z)] and ln(t-z) is real when t-z>0. Using the classical method of contour integration along the branch cut contour D, it can be shown that

$$ {}_0D_z^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_D (t-z)^{-\alpha-1} f(t)dt$$

$$= \frac{\Gamma(\alpha+1)}{2\pi i} [1-\exp\{-2\pi i(\alpha+1)\}] \int_0^z (t-z)^{-\alpha-1} f(t)dt$$

$$= \frac{1}{\Gamma(-\alpha)} \int_0^z (t-z)^{-\alpha-1} f(t)dt \quad (2.1.12)$$

Which agrees with Riemann-Liuville definition (2.1.3) with z=x, and a=0, when α is replaced by -α

Fractional Integration, Fractional Differential Equation using Laplace Transformed Method:

One of the very useful results is formula for Laplace transform of the derivative of an integer order n of a function f(t) is given by

L{ $f^{(n)}(t)$ }= $s^n \bar{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$ (2.2.1)

$= s^n \bar{f}(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0)$ (2.2.2)

$= s^n \bar{f}(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0)$

Where $f^{(n-k)}(0) = c_k$ represents the physically realistic given initial conditions and $\bar{f}(s)$ being the Laplace transform of the function f(t).

Like Laplace transform of integer order derivative, it is easy to shown that the Laplace transform of fractional order derivative is given by

L{ ${}_0D_t^{\alpha}$ f(t)}= $s^{\alpha} \bar{f}(s) - \sum_{k=0}^{n-1} s^k [{}_0D_t^{\alpha-k-1} f(t)]_{t=0}$ (2.2.3)

$= s^{\alpha} \bar{f}(s) - \sum_{k=1}^{n} s^{k-1} c_k$, (2.2.4)

Where n-1 $\le \alpha < n$ and $c_k = \left[{}_0D_t^{\alpha-k} f(t)\right]_{t=0}$ (2.2.5)

Represents the initial conditions which do not have obvious physical interpretation. Consequently, formula (2.2.4) has limited applicability for finding solutions of initial value problem in differential equations.

We now replace α by an integer-order integral J^n and $D^n f(t) \equiv f^{(n)}(t)$ is used to denote the integral order derivative of a function f(t). It turns out that

$D^n J^n$ =I, $J^n D^n \neq$ I. (2.2.6)

This simply means that D^n is the left (not the right inverse) of J^n. It also follows in (2.2.9) with α=n that

$J^n D^n$ f(t) = f(t)- $\sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}$, t>0 (2.2.7)

Similarly, D^{α} can also be defined as the left inverse of J^{α}.We define the fractional derivative of order α>0 with n-1 $\le \alpha < n$ by

${}_0D_t^{\alpha}$ f(t)= $D^n D^{-(n-\alpha)}$ f(t)

= $D^n J^{n-\alpha}$ f(t)

= $D^n [\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau]$ (2.2.8)

On using (2.1.3)

Or, ${}_0D_t^{\alpha}$ f(t)= $\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau$

Where n is an integer and the identity operator 'I' is defined by

$D^0 f(t) = J^0 f(t)$ =If(t)=f(t), so that $D^{\alpha} J^{\alpha}$ =I, α≥0.

Due to the lack of physical interpretation of initial data c_k in (2.2.4), Caputo and Mainardi adopted as an alternative new definition of fractional derivative to solve initial value problems. This new definition was originally introduced by Caputo in the form

${}_0^CD_t^{\alpha}$ f(t)= $J^{n-\alpha} D^n$ f(t)

$= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau$ (2.2.9)

Where n-1 $\le \alpha < n$ and n is an integer.
It follows from (2.2.8) and (2.2.9) that

${}_0D_t^{\alpha}$ f(t)= $D^n J^{n-\alpha}$ f(t) $\neq J^{n-\alpha} D^n$ f(t)= ${}_0^CD_t^{\alpha}$ f(t) (2.2.10)

Unless f(t) and its first (n-1) derivatives vanish at t=0.
Furthermore , it follows (2.2.9) and (2.2.10) that

$J^{\alpha}\, {}_0^CD_t^{\alpha}$ f(t) = $J^{\alpha} J^{n-\alpha} D^n$ f(t)= $J^n D^n$ f(t) = f(t) $- \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}$ (2.2.11)

This implies that ${}_0^CD_t^{\alpha}$ f(t)=

${}_0D_t^{\alpha}$ [f(t) $- \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0)$]

= ${}_0D_t^{\alpha}$ f(t) $- \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0)$ (2.2.12)

This shows that Caputo's fractional derivative incorporates the initial values $f^{(k)}(0)$,

for k=0,1,2,…….,n-1.

The Laplace transform of Caputo's fractional derivative (2.2.12) gives an interesting formula L{ ${}_0^CD_t^{\alpha}$ f(t)}=

$s^{\alpha} \bar{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0) s^{\alpha-k-1}$ (2.2.13)

Transform of $f^{(n)}(t)$ This is a natural generalization of the corresponding well known formula for the Laplace when α=n and can be used to solve the initial value problems in fractional differential equation with physically realistic initial conditions.

Some geometric and physical interpretation of Fractional Calculus is being referred in [26], [30].

Mittag-Leffler function:

The one of the very important function, used in fractional calculus known as Mittag-Leffler function [17], is the generalization of the exponential function e^z. One parameter Mittag-Leffler function is denoted by E_α (z) and is defined by the infinite series,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}\ , \qquad (2.3.1)$$

The two parameter function of this type, which plays a very important role in solving the fractional differential equations, is defined by the infinite series,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}\ , \text{ where } \alpha>0,\ \beta>0 \qquad (2.3.2)$$

It follows from the definition (2.3.2) that

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \qquad (2.3.3)$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{1}{z}(e^z-1) \qquad (2.3.4)$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2}\sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{1}{z^2}(e^z-z-1) \qquad (2.3.5)$$

And in general $E_{1,m}(z) = \frac{1}{z^{m-1}}\{e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}\}$ (2.3.6)

The hyperbolic sine and cosine are also particular cases of the Mittag-Leffler function (2.3.2) as given by

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z) \qquad (2.3.7)$$

$$E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \sinh(z) \qquad (2.3.8)$$

Also we can show that

$$E_{\frac{1}{2},1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{k}{2}+1)} = e^{z^2}\,erfc(-z) \qquad (2.3.9)$$

Where erfc(-z)is the complement of error function defined by

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}dt\ . \qquad (2.3.10)$$

For β=1, we obtain the Mittag-Leffler function in one parameter**:**

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \equiv E_\alpha(z) \qquad (2.3.11)$$

CLASSICAL EOQ MODEL

Notations and Assumptions:

D	Demand rate
Q	Order quantity
U	Per unit cost
C_1	Holding cost per unit
C_3	Set up cost
q(t)	Stock level
T	Ordering interval

w Dual variable of T in geometric programming

In classical EOQ based inventory model, we already have

$$\frac{dq(t)}{dt} = -D\ , \quad \text{for } 0\le t\le T \ = 0, \quad \text{otherwise.} \qquad (3.1)$$

With the initial condition q(0)=Q and with the boundary condition q(T)=0.

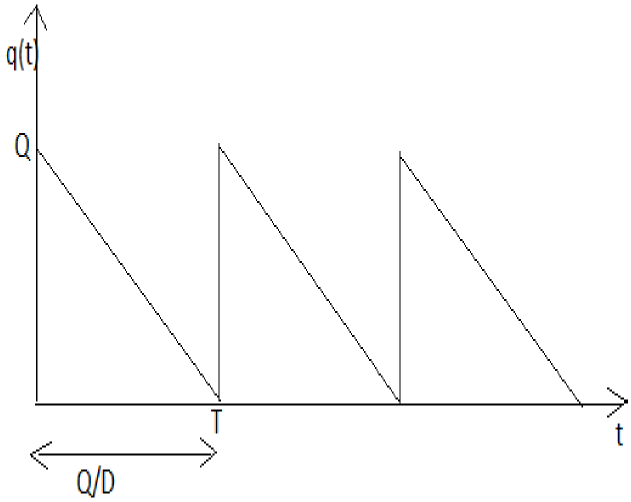

Figure 1.1: Development of inventory level over time.

By solving the equation (3.1), we have q(t)=Q-Dt, for 0≤t≤T (3.2)

And on using the boundary condition q(T)=0, we have Q=DT. (3.3)

Holding cost,

HC(T)=

$$C_1\int_{t=0}^{T} q(t)dt = C_1\int_{t=0}^{T}(Q-Dt)dt = C_1[Qt-\frac{Dt^2}{2}]_{t=0}^{T} = C_1(QT-\frac{DT^2}{2}) = \frac{C_1DT^2}{2} \qquad (3.4)$$

[on using (3.3)]

Total cost, TC(T) =Purchasing cost(PC)+Holding cost(HC)+Set up cost(SC)

$$=UQ + \frac{C_1DT^2}{2} + C_3\,. \qquad (3.5)$$

Total average cost over [0,T] is given by

$$\text{TAC(T)}=\frac{1}{T}[UQ + \frac{C_1DT^2}{2} + C_3]$$

$$= \frac{UQ}{T} + \frac{C_1DT}{2} + \frac{C_3}{T} \qquad (3.6)$$

Then the classical EOQ model is

Min

$$\text{TAC(T)}=\text{UD}+\frac{C_1DT}{2} + \frac{C_3}{T} \qquad (3.7)$$

Subject to, T>0.

Solving (3.7) we can show that TAC(T) will be minimum for

$$T^* = \sqrt{\frac{2C_3D}{C_1}} \qquad (3.8)$$

and $TAC^*(T^*)=UD+\sqrt{2C_1C_3D}$. (3.9)

GENERALIZED EOQ MODEL

We now generalize our discussion by accepting the equation (3.1) as a differential equation of fractional order instead of the linear order. i.e we here consider that demand(D) varies in fractional order say α, here instantaneous inventory level

$$\frac{d^\alpha q(t)}{dt^\alpha}=-D \quad \text{for } 0\le t\le T = 0 \quad \text{otherwise.} \qquad (4.1)$$

With the same initial and boundary condition as described in the previous problem in equation (3.1). i.e q(0)=Q and with q(T)=0. where D is a constant.

Equation (4.1) can be rewritten as ${}_0^C D_t^\alpha$ q(t) = -D for $0\le t\le T$ (4.2)
= 0 otherwise.

Where ${}_0^C D_t^\alpha \equiv J^{1-\alpha} D^1$ is the Caputo fractional derivative as described in (2.2.9) and $D^1 \equiv \frac{d}{dt}$.

To solve the initial value problem of fractional order differential equation (4.2) we apply the Laplace transform method. So taking Laplace transform of the equation (4.2), we have, $L\{{}_0^C D_t^\alpha q(t)\}= -DL\{1\}$

$$\Rightarrow s^\alpha \bar{q}(s)-s^{\alpha-0-1}q(0) = -\frac{D}{s},$$

$\bar{q}(s)$ being Laplace transform of q(t).

$$\Rightarrow s^\alpha \bar{q}(s)= Q\, s^{\alpha-1}-\frac{D}{s}$$

$$\Rightarrow \bar{q}(s)= \frac{Q}{s}-\frac{D}{s^{\alpha+1}}$$

Taking Laplace inversion of above equation we have,

$$q(t) = L^{-1}\{\bar{q}(s)\} = Q-\frac{Dt^\alpha}{\Gamma(\alpha+1)}$$

So the inventory level at any time t based on α ordered decreasing rate of demand

is $q_\alpha(t)= Q-\frac{DT^\alpha}{\Gamma(\alpha+1)}$ for $0\le t\le T$. (4.3)

On using the boundary condition q(T)=0 implies that

$$Q=\frac{DT^\alpha}{\Gamma(\alpha+1)} \qquad (4.4)$$

[for α=1 in (4.3) and (4.4) gives results as in (3.2) and (3.3)].

Generalized Holding Cost:

Now the Holding cost of fractional order, say β i.e.

$$HC_\beta(T)=C_1D^{-\beta}q(t) \qquad (4.1.1)$$

Case1: For α=1 and β=1, Holding cost is

$HC_{1,1}(T)=C_1D^{-1}q(t)=\frac{C_1QT}{2}=\frac{C_1DT^2}{2}$, same as in (3.4).

Case2: For β=1, Holding cost is

$$HC_{1,\alpha}(T)=C_1\int_0^T q_\alpha(t)dt$$

$$=C_1\int_0^T [Q-\frac{Dt^\alpha}{\Gamma(\alpha+1)}]dt$$

$$=C_1[QT-\frac{DT^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}]$$

$$=C_1[QT-\frac{QT.\Gamma(\alpha+1)}{(\alpha+1)\Gamma(\alpha+1)}] \qquad \text{(using (4.4))}$$

$$=\frac{C_1\alpha}{\alpha+1}QT \qquad (4.1.2)$$

Case3: For α=1, Holding cost of order β is

$$HC_{1,\beta}(T)=C_1D^{-\beta}q(t)=C_1\frac{1}{\Gamma(\beta)}\int_0^t (t-x)^{\beta-1}q(x)dx$$

$$=C_1\frac{1}{\Gamma(\beta)}\int_0^t (t-x)^{\beta-1}(Q-Dx)dx$$

$$=C_1\frac{Qt^\beta}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1}d\tau-C_1\frac{Dt^{\beta+1}}{\Gamma(\beta)}\int_0^1 \tau(1-\tau)^{\beta-1}d\tau$$

(putting x = tτ)

$$=C_1\frac{Qt^\beta}{\Gamma(\beta)}[\frac{(1-\tau)^\beta}{(-\beta)}]_{t=0}^1-C_1\frac{Dt^{\beta+1}}{\Gamma(\beta)}B(\beta,2)$$

where B(m,n) is the well known beta function.

$$=C_1\frac{Qt^\beta}{\Gamma(\beta+1)}-C_1\frac{Dt^{\beta+1}}{\Gamma(\beta+2)}$$

For t=T, $HC_{1,\beta}(T)=C_1QT^\beta[\frac{1}{\Gamma(\beta+1)}-\frac{1}{\Gamma(\beta+2)}]$

(using Q=DT for α=1) (4.1.3)

$$=C_1DT^{\beta+1}[\frac{1}{\Gamma(\beta+1)}-\frac{1}{\Gamma(\beta+2)}] \qquad (4.1.4)$$

[The above fractional order integration can also be done by using Laplace transform method. We have to find the fractional order integral

$D^{-\beta}q(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-x)^{\beta-1}q(x)$, where q(x)=Q-Dx.

For this now $L\{D^{-\beta}q(t)\}=L\{D^{-\beta}(Q-Dt)\}=\frac{Q}{s^{\beta+1}}-\frac{D}{s^{\beta+2}}$
(4.1.5)

(Since it is known that

$L\{D^{-\beta}(t^\gamma)\}=s^{-\beta}L\{t^\gamma\}=s^{-\beta}\frac{\Gamma(\gamma+1)}{s^{\gamma+1}}=\frac{\Gamma(\gamma+1)}{s^{\beta+\gamma+1}}$)

Therefore from (4.1.5), we have

$$D^{-\beta}q(t)=L^{-1}\{\frac{Q}{s^{\beta+1}}-\frac{D}{s^{\beta+2}}\}=\frac{Qt^\beta}{\Gamma(\beta+1)}-\frac{Dt^{\beta+1}}{\Gamma(\beta+2)}]$$

Case 4: For any α and β , Holding cost is $HC_{\alpha,\beta}(T)$ $= C_1 D^{-\beta} q_\alpha(t)$,

Where, $q_\alpha(t) = Q - \frac{Dt^\alpha}{\Gamma(\alpha+1)}$

Now $D^{-\beta} q(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-x)^{\beta-1} q(x)$

Then L{ $D^{-\beta} q(t)$ }=L{ $D^{-\beta}(Q - \frac{Dt^\alpha}{\Gamma(\alpha+1)})$}

$= \frac{Q}{s^{\beta+1}} - s^{-\beta} \frac{D}{s^{\alpha+1}} = \frac{Q}{s^{\beta+1}} - \frac{D}{s^{\alpha+\beta+1}}$

Therefore

$D^{-\beta} q(t) =$

$$L^{-1}\{\frac{Q}{s^{\beta+1}} - \frac{D}{s^{\alpha+\beta+1}}\} = \frac{Qt^\beta}{\Gamma(\beta+1)} - \frac{Dt^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

Then for t=T, $HC_{\alpha,\beta}(T)$

$$= C_1 D^{-\beta} q(t) = C_1[\frac{QT^\beta}{\Gamma(\beta+1)} - \frac{DT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}]$$

$$= C_1 QT^\beta[\frac{1}{\Gamma(\beta+1)} - \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}] \quad \text{[using (4.3)]}$$

$$= C_1 \frac{DT^{\alpha+\beta}}{\Gamma(\alpha+1)}[\frac{1}{\Gamma(\beta+1)} - \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}]$$

$$= C_1 DT^{\alpha+\beta}\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} - \frac{1}{\Gamma(\alpha+\beta+1)}\right] \quad (4.1.6)$$

Generalized Total Average Cost:

Total cost(TC)=Purchasing cost(PC)+Holding cost(HC)+Set up cost(SC).

Total Average Cost (TAC)= $\frac{1}{T}$ [Total Cost(TC)]

Case1: For α=1 and β=1, the model is being as our classical EOQ problem where the optimum Total Average Cost $TAC^*_{1,1}(T^*)$ is given in (3.9).

Case2: For any α >0 and β=1,

Here, $TC_{\alpha,1}(T)$ =UQ+ $\frac{C_1\alpha}{\alpha+1} QT + C_3$

Then total average cost $TAC_{\alpha,1}(T) = \frac{1}{T}$[UQ+ $\frac{C_1\alpha}{\alpha+1} QT + C_3$]

$$= \frac{1}{T}[\frac{UDT^\alpha}{\Gamma(\alpha+1)} + \frac{C_1\alpha}{\Gamma(\alpha+2)} DT^{\alpha+1} + C_3]$$

$$= \frac{UDT^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{C_1\alpha}{\Gamma(\alpha+2)} DT^\alpha + \frac{C_3}{T} \quad (4.2.1)$$

Here generalized EOQ model is,

$$\text{Min } TAC_{\alpha,1}(T) = AT^{\alpha-1} + BT^\alpha + \frac{C}{T}, \quad (4.2.2)$$

subject to T ≥0,

where A= $\frac{UD}{\Gamma(\alpha+1)}$, B= $\frac{C_1\alpha D}{\Gamma(\alpha+2)}$ and C= C_3.

(4.2.2) can be taken as a primal geometric programming problem with degree of difficulty (DD) =1.

Dual form of (4.2.2)

$$\text{Max d(w)} = \left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}, \quad (4.2.3)$$

Subject to, $w_1 + w_2 + w_3 = 1$,

(normalized condition) (4.2.4)

w_1(α-1)+ α w_2 - w_3 =0,

(orthogonal condition) (4.2.5)

$w_1, w_2, w_3 \geq 0$.

Primal-dual relations are,

A $T^{\alpha-1} = w_1$ d(w) (4.2.6)

B $T^\alpha = w_2$ d(w) (4.2.7)

$\frac{C}{T} = w_3$ d(w) (4.2.8)

From (4.2.6) and (4.2.7), we have, $\frac{A}{BT} = \frac{w_1}{w_2}$ ⇒T

$$= \left(\frac{A}{B}\right)\left(\frac{w_2}{w_1}\right) \quad (4.2.9)$$

From (4.2.7) and (4.2.8), we have,

$$\left(\frac{B}{C}\right) T^{\alpha+1} = \frac{w_2}{w_3}$$

$$\Rightarrow \frac{B}{C}\left(\frac{Aw_2}{Bw_1}\right)^{\alpha+1} = \frac{w_2}{w_3}$$

$$\Rightarrow \left(\frac{C}{A}\right)\left(\frac{B}{A}\right)^\alpha w_1^{\alpha+1} - w_2^\alpha w_3 = 0 \quad (4.2.10)$$

Now we have to solve for w_1, w_2, w_3 from three system of non linear equations (4.2.4), (4.2.5) and (4.2.10) and obtained the solutions as w_1^*, w_2^* and w_3^* and then from the relation (4.2.9), we will able to obtain T^* for which $TAC_{\alpha,1}(T)$ is minimum. i.e we will able to obtain $TAC^*_{\alpha,1}(T)$ as the minimum of $TAC_{\alpha,1}(T)$ in (4.2.2) and Q*(T) in (4.4).

Case3: For α=1 and for any β, we have the Holding cost , $HC_{1,\beta}(T)$ =

$$C_1 DT^{\beta+1}[\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta+2)}]$$

[from(4.1.4)]

Then Total cost

(TC)=UQ+ $C_1 DT^{\beta+1}[\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta+2)}] + C_3$

$\therefore$ Total average cost $TAC_{1,\beta}(T) = \frac{1}{T}$ {

UQ+ $C_1 DT^{\beta+1}[\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta+2)}] + C_3$ }

$$=UD+ C_1 D\left[\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta+2)}\right]T^{\beta} + \frac{C_3}{T} \quad (4.2.11)$$

[since for α=1, we know that Q=DT.]

$$=A+B\,T^{\beta} + \frac{C}{T} \text{ (say)}$$

Where, A=UD,

$$B= C_1 D\left[\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta+2)}\right], \text{ and } C= C_3. \quad (4.2.12)$$

To minimize $TAC_{1,\beta}(T)$, we again apply geometric programming method.

Let us suppose that M(T)= B $T^{\beta} + \frac{C}{T}$ (4.2.13)

Then the degree of difficulty(DD) in G.P.P(4.2.13)=2-1-1=0

$$\text{Max } d(w)=\left(\frac{B}{w_1}\right)^{w_1}\left(\frac{C}{w_2}\right)^{w_2}$$

Subject to, $w_1 + w_2$ =1 (normalized condition) (4.2.14)

β w_1 - w_2 =0 (orthogonal condition) (4.2.15)

$w_1, w_2 \geq 0$.

Then solving for w_1 and w_2 from the above equation (4.2.14) and (4.2.15), we get

$$w_1 = \frac{1}{\beta+1} \text{ and } w_2 = \frac{\beta}{\beta+1} \quad (4.2.16)$$

Again from the primal-dual relations B $T^{\beta} = w_1$ d(w) and $\frac{C}{T} = w_2$ d(w) , we get

$$\frac{B}{C}T^{\beta+1} = \frac{w_1}{w_2} = \frac{1}{\beta}$$

$$\Rightarrow T^{\beta+1} = \frac{C}{B\beta} \Rightarrow T = \left(\frac{C}{B\beta}\right)^{\frac{1}{\beta+1}} \quad (4.2.17)$$

$\therefore$ we get, Max

$$d(w)=\left(\frac{B}{\frac{1}{\beta+1}}\right)^{\frac{1}{\beta+1}}\left(\frac{C}{\frac{\beta}{\beta+1}}\right)^{\frac{\beta}{\beta+1}}$$

$$= B^{\frac{1}{\beta+1}}C^{\frac{\beta}{\beta+1}}(\beta+1)^{\frac{1}{\beta+1}}\left(\frac{\beta+1}{\beta}\right)^{\frac{\beta}{\beta+1}}$$

$$= B^{\frac{1}{\beta+1}}C^{\frac{\beta}{\beta+1}}\beta^{-\frac{\beta}{\beta+1}}(\beta+1)$$

$\therefore$ Min M(T)=

$$B^{\frac{1}{\beta+1}}C^{\frac{\beta}{\beta+1}}\beta^{-\frac{\beta}{\beta+1}}(\beta+1)$$

$\therefore$ Min $TAC_{1,\beta}(T)$ =

$$TAC^{*}_{1,\beta}(T) = A + B^{\frac{1}{\beta+1}}C^{\frac{\beta}{\beta+1}}\beta^{-\frac{\beta}{\beta+1}}(\beta+1) \quad (4.2.18)$$

Where A, B, C are given in (4.2.12).

Case4: For any α >0 and any β >0, we have the Holding cost as

$$HC_{\alpha,\beta}(T) = C_1 DT^{\alpha+\beta}\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} - \frac{1}{\Gamma(\alpha+\beta+1)}\right]$$

[from(4.1.6)]

Then Total cost $TC_{\alpha,\beta}(T)$

$$=UQ+ C_1 DT^{\alpha+\beta}\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} - \frac{1}{\Gamma(\alpha+\beta+1)}\right] + C_3$$

$\therefore$ Total average cost is given by

$TAC_{\alpha,\beta}(T) = \frac{1}{T}$ {

UQ+ $C_1 DT^{\alpha+\beta}\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} - \frac{1}{\Gamma(\alpha+\beta+1)}\right] +$ C_3 }

$$= \frac{UD}{\Gamma(\alpha+1)}T^{\alpha-1} + C_1 D\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} - \frac{1}{\Gamma(\alpha+\beta+1)}\right]T^{\alpha+\beta-1} + \frac{C_3}{T} \quad (4.2.19)$$

[since we have from (4.4),

Q= $\frac{DT^{\alpha}}{\Gamma(\alpha+1)}$]

$$=A\,T^{\alpha-1} + B\,T^{\alpha+\beta-1} + \frac{C}{T} \quad \text{(say)}$$

Where A= $\frac{UD}{\Gamma(\alpha+1)}$,

B= $C_1 D\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} - \frac{1}{\Gamma(\alpha+\beta+1)}\right]$ and

C= C_3.

Now to minimize $TAC_{\alpha,\beta}(T)$, we apply geometric programming method, and the degree of difficulty(DD) is=3-1-1=1.

$$\text{Max } d(w) = \left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}$$

Subject to, $w_1 + w_2 + w_3$ =1 (normalized condition) (4.2.20)

(α-1) w_1 +(α+β-1) w_2 - w_3 =0(orthogonal condition) (4.2.21)

$w_1, w_2, w_3 \geq 0$.

Again the primal-dual variable relations are given by

A $T^{\alpha-1} = w_1$ d(w) (4.2.22)

$B\,T^{\alpha+\beta-1} = w_2\, d(w)$ (4.2.23)

$\frac{C}{T} = w_3\, d(w)$ (4.2.24)

From (4.2.22) and (4.2.23) we have, $\frac{BT^{\beta}}{A} = \frac{w_2}{w_1}$

$$\Rightarrow T = \left(\frac{Aw_2}{Bw_1}\right)^{\frac{1}{\beta}} \qquad (4.2.25)$$

Again from (4.2.22) and (4.2.24) we have, $\left(\frac{A}{C}\right)T^{\alpha} = \frac{w_1}{w_3}$

$$\Rightarrow \frac{A}{C}\left(\frac{Aw_2}{Bw_1}\right)^{\frac{\alpha}{\beta}} = \frac{w_1}{w_3}$$

$$\Rightarrow \left(\frac{Aw_2}{Bw_1}\right)^{\alpha} = \left(\frac{Cw_1}{Aw_3}\right)^{\beta}$$

$$\Rightarrow \left(\frac{B}{A}\right)^{\alpha}\left(\frac{C}{A}\right)^{\beta} w_1^{\alpha+\beta} - w_2^{\alpha} w_3^{\beta} = 0 \qquad (4.2.26)$$

Now solving for w_1, w_2 and w_3 from three system of non linear equations (4.2.20), (4.2.21) and (4.2.26) we get the solution as w_1^*, w_2^*, w_3^* and hence we calculate Max d(w) i.e $TAC^*_{\alpha,\beta}(T^*)$. Again using (4.2.25), we will able to find T* and hence Q*(T*) by using (4.4).

Numerical example:

A product has a demand of 5000 units per year. The cost of one procurement is Rs 20000 and the holding cost per unit is Rs 100 per year. The replacement is instantaneous and no shortages are allowed. We shall now calculate for different values of α and β,

(a). The economic lot size,(EOQ)
(b). Optimal total average cost,
(c). Optimal time period.

Where the set up cost is Rs.10000.
This is given in terms of some tables and figures.

Table-1: Optimum value of T*, $TAC^*_{\alpha,1}(T^*)$ and Q^* (T*) for different vales α and β=1.0.

α	**A=** $\frac{UD}{\Gamma(\alpha+1)}$ **, B=** $\frac{C_1\alpha D}{\Gamma(\alpha+2)}$ **, C=** C_3 **=10000.**	**T***	$TAC^*_{\alpha,1}(T^*)$	**Q*(T*)**
0.1	A=105114000, B=47779	19800	142778	14135
0.2	A=108912000, B=90760.4	4799.98	618094	29668.5
0.3	A=111424000, B=128566	2022.22	1.80209e+6	54663.6
0.4	A=112706000, B=161009	1050	4.33685e+6	91073.7
0.5	A=112838000, B=188063	600.007	9.2131e+6	138198
0.6	A=111917000, B=209845	355.554	1.77985e+7	189850
0.7	A=110055000, B=226583	208.164	3.16949e+7	230921
0.8	A=107367000, B=238594	112.501	5.21835e+7	234837
0.9	A=103975000, B=246258	46.913	7.8625e+7	165983
1.0	A=100000000, B=250000	0.2	1.001e+8	1000

Above table shows optimal results of total average cost, time period and order quantity for different α. It is seen that as α increases T^* decreases [Fig-2] and TAC*(T*) increases [Fig-3] but Q*(T*) increases up to certain value of α and then decrease [Fig-4].

Table-2: Optimum values of T*, $TAC^*_{1,\beta}(T^*)$ & Q*(T*) for different values β and α=1.0.

β	A=UD, B= $C_1 D\left[\frac{1}{\Gamma(\beta+1)} - \frac{1}{\Gamma(\beta+2)}\right]$ C=C_3=10000	T*	$TAC^*_{1,\beta}(T^*)$	Q*(T*)
0.1	A=100000000, B=4779	15.8709	100007000	79354.5
0.2	A=100000000, B= 90760.4	0.608454	100099000	3042.27
0.3	A=100000000, B= 128566	0.35403	100122000	1770.15
0.4	A=100000000, B= 161009	0.264367	100132000	1321.84
0.5	A=100000000, B= 188063	0.224466	100134000	1122.33
0.6	A=100000000, B= 209845	0.205338	100130000	1026.69
0.7	A=100000000, B= 226583	0.196755	100123000	983.775
0.8	A=100000000, B= 238594	0.1943	100116000	971.5
0.9	A=100000000, B= 246258	0.195782	100108000	978.91
1.0	A=100000000, B= 250000	0.2	100100000	1000

Above table shows optimal results of total average cost, time period and order quantity for different β when α is being fixed as 1.0. It is seen that as β increases T^*decreases [Fig-5] and TAC*(T*) increases up to certain values of β and after then it decreases [Fig-6] but Q*(T*) decreases [Fig-7] respectively.

TABLE- 3, For α=0.2 and any β Optimum value of T*,TAC*(T*),Q*(T*)

β	A= $\frac{UD}{\Gamma(\alpha+1)}$ **B=** $C_1D\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}-\frac{1}{\Gamma(\alpha+\beta+1)}\right]$ C= C_3	T*	TAC*(T*)	Q*(T*)
0.1	A=108912000, B=15288.2	0.3061442 e+14	0.001777024	2.71164e+6
0.2	A=108912000, B=29565.8	0.31727 82e +14	0.001955948	2.73109e+6
0.3	A=108912000, B=42584.8	0.7903284e+14	0.005619694	3.278e+6
0.4	A=108912000, B=54167.4	0.5234629e+15	0.07035799	4.78437e+6
0.5	A=108912000, B=64199	0.7001797e+16	1.132333	8.03692e+6
0.6	A=108912000, B=72624.7	0.1308728e+18	27.39773	1.44350e+7
0.7	A=108912000, B=79439.5	0.2608468e+19	1143.926	2.62618e+7
0.8	A=108912000, B=84680.8	0.30775252e+14	84680.8	2.71409e+6
0.9	A=108912000, B=88421.3	27376.76	276340.4	42026.6
1.0	A=108912000, B=90760.4	4800.080	618095.7	29688.6

Above table shows optimal results of total average cost, time period and order quantity for different β and fixed α=0.2. It is seen that as β increases T^*decreases and TAC*(T*) increases but Q*(T*) decreases.

TABLE- 4, For α=0.4 and any β Optimum value of T*,TAC*(T*),Q*(T*)

β	A= $\frac{UD}{\Gamma(\alpha+1)}$ **B=** $C_1D\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}-\frac{1}{\Gamma(\alpha+\beta+1)}\right]$ C=C_3.	T*	TAC*(T*)	Q*(T*)
0.1	A=112706000, B=28157.9	0.99e+20	0.0001162177	5.61269e+11
0.2	A=112706000, B=54167.1	0.99e+20	0.0006572407	5.61269e+11
0.3	A=112706000, B=77635.7	0.99e+20	0.07798352	5.61269e+11
0.4	A=112706000, B=98297	0.99e+20	9.849592	5.61269e+11
0.5	A=112706000, B=115999	0.99e+20	1161.157	5.61269e+11
0.6	A=112706000, B=130689	0.99e+20	130698	5.61269e+11
0.7	A=112706000, B=142402	178756.2	556786.8	710922
0.8	A=112706000, B=151244	15372.32	1386643	266448
0.9	A=112706000, B=157378	3211.487	2661012	142429
1.0	A=112706000, B=161009	1050.007	4336857	91073.9

Above table shows optimal results of total average cost, time period and order quantity for different β and fixed α=0.4. It is seen that as β increases T^*decreases and TAC*(T*) increases but Q*(T*) decreases.

TABLE- 5, For α=0.6 and any β Optimum value of T*,TAC*(T*),Q*(T)

β	A= $\frac{UD}{\Gamma(\alpha+1)}$ **B=** $C_1D\left[\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}-\frac{1}{\Gamma(\alpha+\beta+1)}\right]$, C= C_3	T*	TAC*(T*)	Q*(T*)
0.1	A=111917000, B=37929.4	0.99e+20	1.161722	5.56223e+15
0.2	A=111917000, B=72624.7	0.99e+20	8.400761	5.56223e+15
0.3	A=111917000, B=103639	0.99e+20	1038.556	5.56223e+15
0.4	A=111917000, B=130689	0.99e+20	130690.1	5.56223e+15
0.5	A=111917000, B=153637	8490250	946886.4	8.03934e+7
0.6	A=111917000, B=172474	154408.4	2821948	7262240
0.7	A=111917000, B=187298	13954.59	5740983	1716720
0.8	A=111917000, B=198291	2751.011	9421709	647981
0.9	A=111917000, B=205707	854.9325	13533090	321386
1.0	A=111917000, B=209845	355.5568	17798480	189851

Above table shows optimal results of total average cost, time period and order quantity for different β and fixed α=0.6. It is seen that as β increases T^*decreases and TAC*(T*) increases but Q*(T*) decreases.

TABLE- 6, For α=0.8 and any β Optimum value of T*,TAC*(T*),Q*(T*)

β	A= $\dfrac{UD}{\Gamma(\alpha+1)}$ **B=** $C_1 D\left[\dfrac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}-\dfrac{1}{\Gamma(\alpha+\beta+1)}\right]$ C= C_3	T*	TAC*(T)	Q*(T)
0.1	A=107367000, B=44410.7	0.99e+20	11202.86	5.32537e+19
0.2	A=107367000, B=84680.8	0.99e+20	95439.10	5.32537e+19
0.3	A=107367000, B=120376	0.6884457e+11	2189869	2.51268e+12
0.4	A=107367000, B=151244	0.1342708e+8	8059433	2.70537e+9
0.5	A=107367000, B=177199	163168.6	0.1622528e+8	7.94238e+7
0.6	A=107367000, B=198291	11330	0.2489523e+8	9.40211e+6
0.7	A=107367000, B=214687	1937.859	0.3307765e+8	2.28928e+6
0.8	A=107367000, B=226643	559.7714	0.4038387e+8	847715
0.9	A=107367000, B=234487	224.8403	0.4673492e+8	408640
1.0	A=107367000, B=238594	112.5009	0.5218568e+8	234837

Above table shows optimal results of total average cost, time period and order quantity for different β and fixed α=0.8. It is seen that as β increases T^*decreases and TAC*(T*) increases but Q*(T*) decreases.

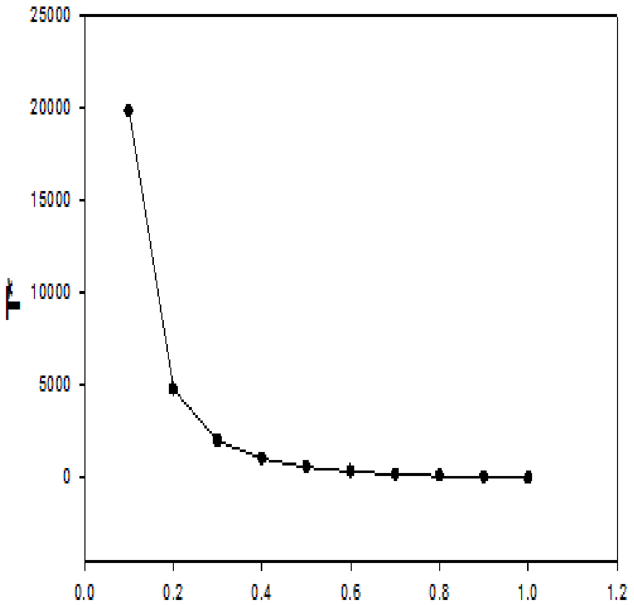

Figure-2: Rough sketch of α versus T* graph for fixed β=1.0

We see from the above figure that for fixed β=1, as α increases T* decreases and this decreasing rate is very high when α varies between 0.1 to 0.4. The decreasing rate of T* becomes slow after 0.4.

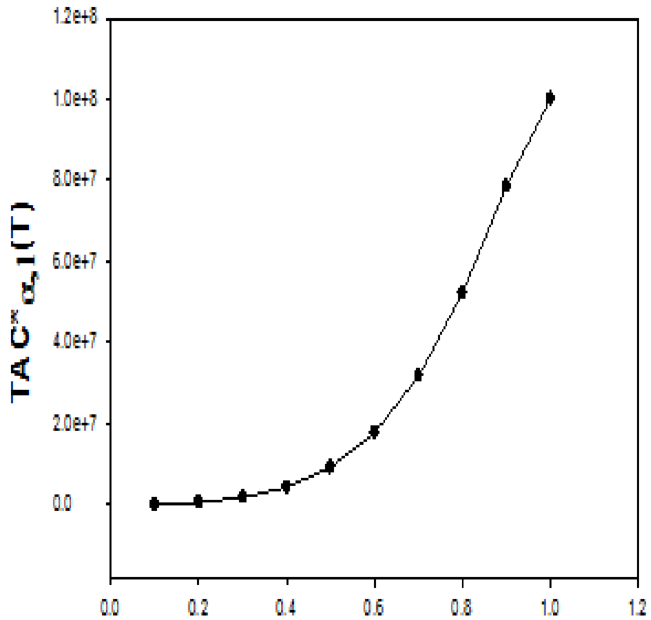

Figure-3: Rough sketch of α versus $TAC^*_{\alpha,1}(T^*)$ graph for fixed β=1.0

We see from the above figure that as α increases, $TAC^*_{\alpha,1}(T)$ increases and this increasing rate very high when $\alpha \geq 0.5$.

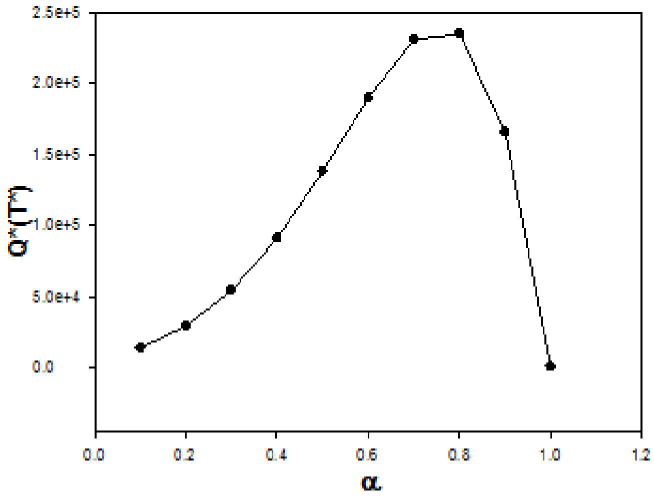

Figure-4: Rough sketch of α versus Q*(T*) graph for fixed β=1.0

The above figure shows that as α increase up to 0.8, Q*(T*) increases, but when α>0.8, Q*(T*) decreases very highly.

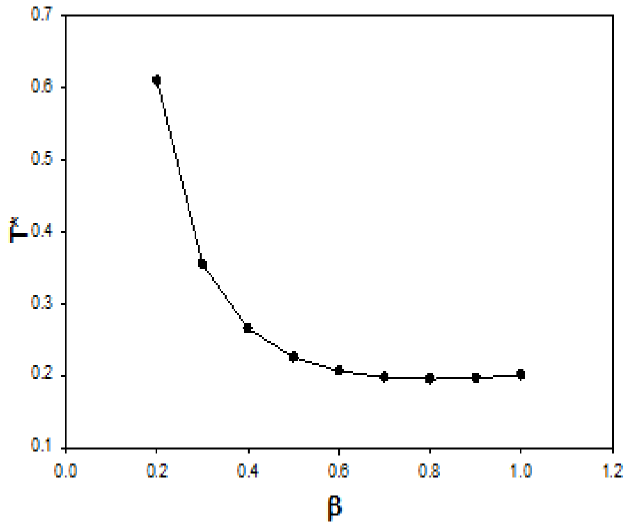

Figure-5: Rough sketch of β versus T* graph for fixed α=1.0

We see from the above figure that as β increases , T* decreases. The decreasing rate is very high when β is in between 0.1 and 0.5.

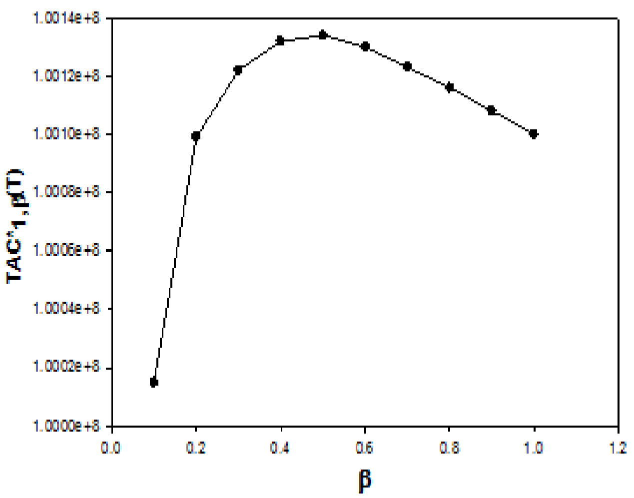

Figure-6: Rough sketch of β versus $TAC^*_{1,\beta}(T^*)$ graph for fixed α=1.0

We see from the above figure that as β increases up to 0.5, $TAC^*_{1,\beta}(T^*)$ increases very highly and when β>0.5, $TAC^*_{1,\beta}(T^*)$ decrease slowly.

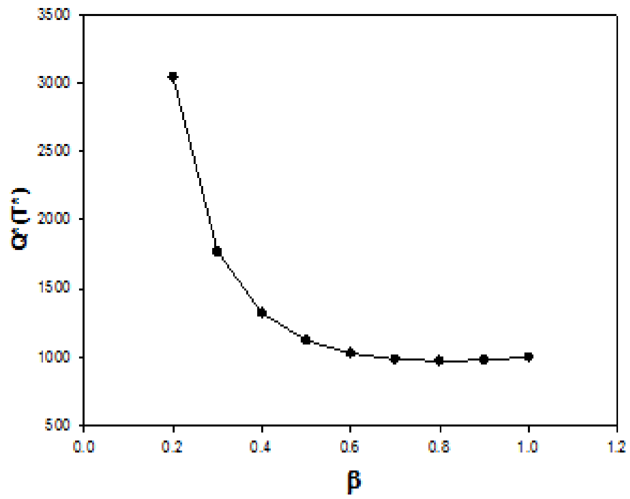

Figure-7: Rough sketch of β versus Q*(T*) graph for fixed α=1.0

The above figure shows that that as β increases, Q*(T*) decreases. The decreasing rate is very high when β is in between 0.1 and 0.5.

CONCLUSION

Fractional Calculus is the generalization of the ordinary calculus. In this feature article, we have briefly shown some of the role and application of the fractional in our well known classical EOQ model of inventory control in operation research so it can be made more general. It is shown that classical holding cost (HC) total average cost(TAC) are the particular case of our generalized holding cost and total average cost which is based on the fractional order integration and differentiation.

Although the fractional order calculus is a 300-years old topic, only very rare application is studied in any operation research model. Still, ordinary calculus is much more familiar and more preferred, may be because its applications are more apparent. However it is expected that this new branch of applied mathematics will able to fill the gap between ordinary calculus and fractional order calculus. Fractional calculus has the potentiality of useful application in any operation research model. In future it would be a very useful tool to describe any operation research model more precisely.

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