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On Representation of Functions in $L^2(0,1)$ By Using Affine System of Walsh –Paley System Type

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ABSTRACT: In this paper , we introduced a notion of affine system of Walsh-Paley system type .We considerbiorthogonal series of the form :$\boldsymbol{f} = \sum_{n=0}^{\infty}(\boldsymbol{f}, \Psi_n)\boldsymbol{\varphi_n}$,where , $\boldsymbol{f}$ belongs to space $L^2(0,1) = H$, (H is a Hilbert space) , which we are represented by :$\boldsymbol{H = E \oplus W_0H \oplus W_1H}$,where , $\boldsymbol{E} = \boldsymbol{span}\ (\boldsymbol{w})$, ($\boldsymbol{w}$is Walsh-Paley function) , and $\boldsymbol{W_0}$, $\boldsymbol{W_1}$are two operators which are defined . Some properties are given with proves for this representation , as well as we gave a general forms for this space with its prove by using induction rule . Also $\{\boldsymbol{\varphi_n}\}_{n\geq 0}$ is the affine system of Walsh –Paley system type of a function $\boldsymbol{\varphi}$. We introduced this function by using Walsh-Paley function . Also ,$(\boldsymbol{f}, \Psi_n)$are the Fourier coefficients of a function $\boldsymbol{f} \in L^2(0,1)$ in the Walsh -Paley system . We are proving that these coefficients formsbiorthogonal conjugate to the system $\{\boldsymbol{\varphi_n}\}_{n\geq 0}$ of a function $\boldsymbol{\varphi}$. Finally , we showed that , the affine system of Walsh –Paley system type of a function $\boldsymbol{\varphi}$ is Bessel system.

KEYWORDS: Biorthogonal series,The space $L^2(0,1)$, The Fourier - Walsh series , Walsh - Paley system , Bessel system.

I. INTRODUCTION

Definition (1) : Suppose that the function$\boldsymbol{\varphi} \in L^2(0,1)$,and$\int_0^1 \boldsymbol{\varphi(t)dt} = 0$ .For $n \in N \cup \{0\}$, with regard to the standard representation $n = 2^k + j$, we set :

$\boldsymbol{\varphi_n = \varphi_{k,j} = \varphi_\alpha = W^\propto \varphi = W_{\alpha_1} \ \ldots W_{\alpha_k} \varphi}$, $\boldsymbol{k = 0, 1, \ldots}$, $\boldsymbol{j = 0, \ldots}, 2^k - 1$

Besides , we set $\boldsymbol{\varphi_0(t) \equiv 1}$. The system $\{\boldsymbol{\varphi_n}\}_{n\geq 0} = \{\boldsymbol{W^\propto \varphi}\}_{\propto \in \Omega}$is the affine system of Walsh type of the function $\boldsymbol{\varphi}$without the constant $\boldsymbol{\varphi_0(t) \equiv 1}$, where

$\boldsymbol{W^\propto = W_{\alpha_1}} \ \ . \ \ . \ \ .\boldsymbol{W_{\alpha_k}}$, $\alpha = (\alpha_1 , \ . \ . \ . \ . , \alpha_k) \in \Omega = \cup_{k=0}^{\infty}\{0,1\}^k$.

Denote the product of the operators : the operator $\boldsymbol{W_{\alpha_k}}$acts first , $\boldsymbol{W_{\alpha_1}}$actslast , and the empty product is set equal to the identity operator I .

Definition (2) : The Walsh - Paley system [3], $\boldsymbol{w} = (\boldsymbol{w_n}, n \in N \cup \{0\}$) is defined as the product of Rademacher functions in the following way . if$n = \sum_{k=0}^{\infty} n_k 2^k \in N \cup \{0\}$ has binary coefficients (n_k , $k \in N \cup \{0\}$) , then :$\boldsymbol{w_n} = \prod_{k=0}^{\infty} \boldsymbol{r_k^{n_k}}$,where ,

$$\boldsymbol{r}(\boldsymbol{x}) = \begin{cases} \boldsymbol{1} & , \quad \boldsymbol{x} \in (\boldsymbol{0}, \ \boldsymbol{1/2}) \\ \boldsymbol{-1} & , \quad \boldsymbol{x} \in (\boldsymbol{1/2}, \ \boldsymbol{1}) \end{cases}$$

$\boldsymbol{r(x+k) = r(x)}$, $\boldsymbol{x \in (0, 1)}$, $k \in N$, and,$\boldsymbol{r_k(x) = r(2^k x)}$, $\boldsymbol{x \in R}$, $k \in N$

It is easy to show that the Walsh system $\{\boldsymbol{w_n}\}_{n\geq 0}$ is affine system of Walsh type , generating by :$\boldsymbol{w} = \boldsymbol{\chi}\,[\boldsymbol{0}, \boldsymbol{1/2}) - \boldsymbol{\chi}\,[\boldsymbol{1/2}, \boldsymbol{1})$.

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Definition (3) : The system of functions $\{\Psi_{\boldsymbol{n}}\}_{n\geq 0}$ can be defined by the equalities:
$\Psi_{\boldsymbol{n}} = \Psi_{\alpha} = \sum_{v=0}^{k} y(\alpha_{v+1}, . . . , \alpha_k) w(\alpha_1, . . . , \alpha_v)$, $n \in N$,where , $y(\alpha_{v+1}, . . . , \alpha_k)$ are computed in Sec. 3 ,andbesides , put $\Psi_{\boldsymbol{0}}(\boldsymbol{t}) \equiv \boldsymbol{1}$.
Consider thebiorthogonal expansion :

$$\boldsymbol{f} \sim \sum_{\boldsymbol{n=0}}^{\infty} (\boldsymbol{f}, \Psi_{\boldsymbol{n}}) \boldsymbol{\varphi}_{\boldsymbol{n}} . \quad . \quad . \quad (1)$$

of a function $\boldsymbol{f} \in L^2(0,1)$ in the system $\{\boldsymbol{\varphi}_{\boldsymbol{n}}\}_{n\geq 0}$.
The main question of the present paper , first to show that the system $\{\boldsymbol{\varphi}_{k,j}\}_{j=0}^{2^{k-1}}$ (k - fixed) is orthogonal block . Second to show that the system $\{\Psi_{\boldsymbol{n}}\}_{n\geq 0}$ is biorthogonal conjugate to the system $\{\boldsymbol{\varphi}_{\boldsymbol{n}}\}_{n\geq 0}$ of a function $\boldsymbol{\varphi}$. The main question under consideration is closely related to wavelet theory , in particular , to periodic wavelets . In the classical monographs of Daubechies [2] , periodic wavelets as periodized wavelets in $L^2(R)$ were constructed . General periodic wavelets and periodic a multiple - scale analysis were studied in the paper of Chui and Wang [1] , Skopina [4] .

II. NOMATION AND AUXILIARY STATEMENTS

* $\Omega = \bigcup_{k=0}^{\infty} \{0,1\}^k$, the family of all finite sequences $\alpha = (\alpha_1, . . . , \alpha_k)$ consisting of zeros and ones (including the empty sequence for $k = o$) ;
* $|\alpha|$, the length of a sequence $\alpha \in \Omega$, i.e. , $|\alpha| = k$ for $\alpha = (\alpha_1, . . . , \alpha_k)$ (the length of an empty sequence is set to zero) ;
*$\alpha\beta$, the concatenation of sequences $\alpha, \beta \in \Omega$: if $\alpha = (\alpha_1, . . . , \alpha_k)$ and $\beta = (\beta_1, . . . , \beta_l)$, then $\alpha\beta = (\alpha_1, . . . , \alpha_k, \beta_1, . . . , \beta_l)$.
Let us point out the natural one - to - one correspondence between the set of natural numbers $N$and the family Ω . Suppose that$n \in N$ and $n = 2^k + j$ is the standard representation . Consider the binary expansion :$j = \sum_{v=1}^{k} \alpha_v 2^{k-v}$of the number $j = 0$, . . . , $2^k - 1$ .The collection$\alpha = (\alpha_1, . . . , \alpha_k) \in \Omega$.Is assigned to natural number n .
Terekhin , [5] , consider the operator structure of multi translation $\{V_0, V_1\}$, setting

$$V_0 f(t) = \mathbf{2}^{1/2} f(2t) \quad , \quad V_1 f(t) = \mathbf{2}^{1/2} f(2t-1)$$

In our notation , we will transform $\{V_0, V_1\}$ to our operators $\{W_0, W_1\}$ as follow :

$$W_0 f(t) = \frac{\boldsymbol{V_0 f(t) + V_1 f(t)}}{\sqrt{\mathbf{2}}} = \frac{\mathbf{2}^{1/2} \boldsymbol{f}(\mathbf{2}\boldsymbol{t}) + \mathbf{2}^{1/2} \boldsymbol{f}(\mathbf{2}\boldsymbol{t} - \mathbf{1})}{\sqrt{\mathbf{2}}} = \frac{\mathbf{2}^{\frac{1}{2}}(\boldsymbol{f}(\mathbf{2}\boldsymbol{t}) + \boldsymbol{f}(\mathbf{2}\boldsymbol{t} - \mathbf{1}))}{\sqrt{\mathbf{2}}} = \boldsymbol{f}(\mathbf{2}\boldsymbol{t}) + \boldsymbol{f}(\mathbf{2}\boldsymbol{t} - \mathbf{1})$$

$$W_1 f(t) = \frac{\boldsymbol{V_0 f(t) - V_1 f(t)}}{\sqrt{\mathbf{2}}} = \frac{\mathbf{2}^{1/2} \boldsymbol{f}(\mathbf{2}\boldsymbol{t}) - \mathbf{2}^{1/2} \boldsymbol{f}(\mathbf{2}\boldsymbol{t} - \mathbf{1})}{\sqrt{\mathbf{2}}} = \frac{\mathbf{2}^{\frac{1}{2}}(\boldsymbol{f}(\mathbf{2}\boldsymbol{t}) - \boldsymbol{f}(\mathbf{2}\boldsymbol{t} - \mathbf{1}))}{\sqrt{\mathbf{2}}} = \boldsymbol{f}(\mathbf{2}\boldsymbol{t}) - \boldsymbol{f}(\mathbf{2}\boldsymbol{t} - \mathbf{1})$$

Now , we introduce $\boldsymbol{H}$as follow :

$$\mathbf{H} = \mathbf{E} \oplus \mathbf{W_0 H} \oplus \mathbf{W_1 H} \quad . \quad . \quad . \quad (2)$$

Lemma (1) : Two operators W_0, W_1 are satisfying the following properties :

1. $(W_0 f, W_1 g) = 0 \forall f, g \in \boldsymbol{H}$.
2. W_0, W_1 are two isometric operators .
3. $(W_0 f, W_0 g) = (f, g) \forall f, g \in \boldsymbol{H}$.
4. $(W_1 f, W_1 g) = (f, g) \forall f, g \in \boldsymbol{H}$.

Prove (1) :we want to show that $(W_0 f, W_1 g) = 0 \forall f, g \in \boldsymbol{H}$ .This is shown as :

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$$(W_0 f\,,\quad W_1\, g\,) = \int_0^1 (\boldsymbol{f(2t)} + \boldsymbol{f(2t-1)}\,)\,\big(\,\boldsymbol{g(2t)} - \boldsymbol{g(2t-1)}\big)\ \boldsymbol{dt}$$

$$= \int_0^1 \boldsymbol{f(2t)\ g(2t)\ dt}$$

$$- \int_0^1 \boldsymbol{f(2t)\ g(2t-1)\ \ dt} + \int_0^1 \boldsymbol{f(2t-1)\ g(2t)\ dt} - \int_0^1 \boldsymbol{f(2t-1)\ g(2t-1)\ \ dt}$$

The integral $\int_0^1 \boldsymbol{f(2t)\ g(2t)\ dt}$ can be computed by :Supp $\boldsymbol{f(2t)} \subset [\,0,\ \frac{1}{2}]$ and Supp $\boldsymbol{g(2t)} \subset [\,0,\ \frac{1}{2}]$, then , we have:Supp $\boldsymbol{f(2t)\ g(2t)} \subseteq$ Supp $\boldsymbol{f(2t)}$ $\cap$ Supp $\boldsymbol{g(2t)} = [\,0,\ \frac{1}{2}]$, now let $q = 2t$,when $t = 0$, we have $q = 2(0) = 0$, and , when $t = 1$, we have $q = 2(1) = 2$.Then , the integral $\int_0^1 \boldsymbol{f(2t)\ g(2t)\ dt} =$ *1202**fq gq dq=** 1201**fq gq dq*** , and , Supp ***fq gq***$\subset$ *0, 1 .*

Now , we want to computed the integration $\int_0^1 \boldsymbol{f(2t)\ g(2t-1)\ \ dt}$ as follow :Since : Supp $\boldsymbol{f(2t)} \subset [\,0,\ \frac{1}{2}]$ and Supp $\boldsymbol{g(2t-1)} \subset [\frac{1}{2},\ 1\,]$ then , we have :Supp $\boldsymbol{f(2t)g(2t-1)} \subseteq$ Supp $\boldsymbol{f(2t)}$ $\cap$ Supp $\boldsymbol{g(2t-1)} = \emptyset$, then , we have : $\int_0^1 \boldsymbol{f(2t)\ g(2t-1)\ \ dt = 0}$.

For integral $\int_0^1 \boldsymbol{f(2t-1)\ g(2t)\ dt}$, we obtain :supp$\boldsymbol{f(2t-1)} \subset [\frac{1}{2},\ 1\,]$, and , Supp $\boldsymbol{g(2t)} \subset [\,0,\ \frac{1}{2}]$, then :Supp $\boldsymbol{f(2t-1)g(2t)} \subseteq$ Supp $\boldsymbol{f(2t-1)}$ $\cap$ Supp $\boldsymbol{g(2t)} = \emptyset$,There for $\int_0^1 \boldsymbol{f(2t-1)\ g(2t)\ \ dt = 0}$.

We want to compute a last integration $\int_0^1 \boldsymbol{f(2t-1)\ g(2t-1)\ \ dt}$:Supp $\boldsymbol{f(2t-1)} \subset [\frac{1}{2},\ 1\,]$, and , Supp $\boldsymbol{g(2t-1)} \subset [\frac{1}{2},\ 1\,]$, There for Supp $\boldsymbol{f(2t-1)g(2t-1)} \subseteq$ Supp $\boldsymbol{f(2t-1)}$ $\cap$ Supp $\boldsymbol{g(2t-1)} = \left[\frac{1}{2},\ 1\right]$.Let $q = 2t - 1$, when $t = 0$, we have $q = 2(0) - 1 = 0 - 1 = -1$, also , when $t = 1$, we have $q = 2(1) - 1 = 2 - 1 = 1$, there for : $\int_0^1 \boldsymbol{f(2t-1)\ g(2t-1)\ \ dt} = \frac{1}{2}\int_{-1}^1 \boldsymbol{f(q)\ g(q)\ \ dt} = \frac{1}{2}\int_0^1 \boldsymbol{f(q)\ g(q)\ \ dt}$, Supp $\boldsymbol{f(q)g(q)} \subset [\,0,\ \ 1\,]$.

Byusing compensation, we have:$(\,W_0\, f\,,\ \ W_1\, g\,) = \frac{1}{2}\int_0^1 \boldsymbol{f(q)\ g(q)\ dq}$ - $0 + 0 - \frac{1}{2}\int_0^1 \boldsymbol{f(q)\ g(q)\ dq}\ = 0$.

<u>**Prove (2) :**</u>we want to show that $W_0\,,\, W_1$ are two isometric operators :That is means we are going to prove that $\|\, W_0\, f\| = \|f\|$:

$$\|\, W_0\, f\|^2 = \int_0^1 (\boldsymbol{f(2t)} + \boldsymbol{f(2t-1)})^{\boldsymbol{2}}\boldsymbol{dt} = \int_0^1 \big(f^2(2t) + 2f(2t)f(2t-1) + f^2(2t-1)\big)\, dt$$

$$= \int_0^1 f^2(2t)dt + 2\int_0^1 f(2t)f(2t-1)dt + \int_0^1 f^2(2t-1)dt$$

Let us compute the integral $\int_0^1 f^2(2t)dt$ as :Supp$\boldsymbol{f^2(2t)} \subset [\,0,\ \frac{1}{2}]$, let $q = 2t$, when $t = 0$, we have $q = 2(0) = 0$, and , when $t = 1$, we have $q = 2(1) = 2$.

There for , $\int_0^1 f^2(2t)dt = \frac{1}{2}\int_0^2 f^2(q)dq = \frac{1}{2}\int_0^1 f^2(q)dq$

Also the integration $\int_0^1 f(2t)f(2t-1)dt$ can be computed as :Since , supp$\boldsymbol{f(2t)} \subset [0,\ \frac{1}{2}]$, and , supp$\boldsymbol{f(2t-1)} \subset [\frac{1}{2},\ 1]$, then:supp$\boldsymbol{f(2t)f(2t-1)} \subseteq$ supp $\boldsymbol{f(2t)} \cap$ supp $\boldsymbol{f(2t-1)} = \emptyset$, then :$\int_0^1 f(2t)f(2t-$ *1dt=0.*

Now , a last integration $\int_0^1 f^2(2t-1)dt$ can be found by :Supp$\boldsymbol{f^2(2t-1)} \subset [\frac{1}{2},\ 1\,]$ and Let $q = 2t - 1$, when $t = 0$, we have $q = 2(0) - 1 = 0 - 1 = -1$, also , when $t = 1$, we have $q = 2(1) - 1 = 2 - 1 = 1$, there for : $\int_0^1 f^2(2t-1)dt = \frac{1}{2}\int_{-1}^1 f^2(q)dq = \frac{1}{2}\int_0^1 f^2(q)dq$, There for :$\|\, W_0\, f\|^2 = \frac{1}{2}\int_0^1 f^2(q)dq + 0 + \frac{1}{2}\int_0^1 f^2(q)dq = \int_0^1 f^2(q)dq = \|f\|^2$,$\|\, W_0\, f\| = \|f\| \rightarrow W_0$ is isometric operator .By using the same working from above we see that : W_1 is isometric operator .

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Prove (3) : From property (2) , we have :$\| W_0(f+g)\|^2 = \|f+g\|^2 \rightarrow \| W_0 f\|^2 + 2(W_0 f , W_0 g) + \| W_0 g\|^2 = \|f\|^2 + 2(f , g) + \|g\|^2$,Since , $\| W_0 f\|^2 = \|f\|^2$, and , $\| W_0 g\|^2 = \|g\|^2$, then we have :$2(W_0 f , W_0 g) = 2(f , g) \rightarrow (W_0 f , W_0 g) = (f , g) \forall f , g \in \boldsymbol{H}$.

Prove (4) : From property (2) , we have :$\| W_1(f+g)\|^2 = \|f+g\|^2$

$$\| W_1 f\|^2 + 2(W_1 f , W_1 g) + \| W_1 g\|^2 = \|f\|^2 + 2(f , g) + \|g\|^2$$

Since , $\| W_1 f\|^2 = \|f\|^2$, and , $\| W_1 g\|^2 = \|g\|^2$, then we have :$2(W_1 f , W_1 g) = 2(f , g) \rightarrow (W_1 f , W_1 g) = (f , g) \forall f , g \in \boldsymbol{H}$.

Lemma (1) , implies this decomposition :

$$\boldsymbol{H} = \boldsymbol{E} \oplus \boldsymbol{W_0 H} \oplus \boldsymbol{W_1 H} \quad . \quad . \quad . \quad (2)$$

Acting to eq. (2) by isometric operators $\boldsymbol{W_0}$ and $\boldsymbol{W_1}$we put :

$$\boldsymbol{W_0 H} = \boldsymbol{W_0 E} \oplus \boldsymbol{W_0}^2 \boldsymbol{H} \oplus \boldsymbol{W_0 W_1 H}. \quad . \quad . \quad (3)$$

$$\boldsymbol{W_1 H} = \boldsymbol{W_1 E} \oplus \boldsymbol{W_1 W_0 H} \oplus \boldsymbol{W_1}^2 \boldsymbol{H}$$

The eq. (2) will be :$\boldsymbol{H} = \boldsymbol{E} \oplus \boldsymbol{W_0 H} \oplus \boldsymbol{W_1 H} \oplus (\oplus_{|\alpha|=2} \boldsymbol{W}^{\propto} \boldsymbol{H}). \quad . \quad . \quad (4)$

In general case :

Lemma (2):$\boldsymbol{H} = \left(\oplus_{|\alpha|<k} \boldsymbol{W}^{\propto}\boldsymbol{E}\right) \oplus (\oplus_{|\alpha|=k} \boldsymbol{W}^{\propto}\boldsymbol{H}). \quad . \quad . \quad (5)$

Proof : Eq.(5) can be proved by using induction rule : First , we assume that eq.(5) is true when $k=0$: $\boldsymbol{H} = \boldsymbol{W}^{\mathbf{0}}\boldsymbol{H} = \boldsymbol{H}$, where $\boldsymbol{W}^{\mathbf{0}} = I$ .Second , we assume that it is true when $k=s$:Finally , we want to prove it is true when $k = s+1$:

Multiplyingeq.(5) by

$$:\boldsymbol{W_0 H} = \left(\oplus_{|\alpha|<s} \boldsymbol{W_0}(\boldsymbol{W}^{\propto}\boldsymbol{E})\right) \oplus (\oplus_{|\alpha|=s} \boldsymbol{W_0}(\boldsymbol{W}^{\propto}\boldsymbol{H})) = (\oplus_{\substack{\mathbf{0}<|\alpha|<s+1 \\ \alpha_1=0}} \boldsymbol{W}^{\propto}\boldsymbol{E}) \oplus (\oplus_{\substack{|\alpha|=s+1 \\ \alpha_1=0}} \boldsymbol{W}^{\propto}\boldsymbol{H})$$

$$\boldsymbol{W_1 H} = \left(\oplus_{|\alpha|<s} \boldsymbol{W_1}(\boldsymbol{W}^{\propto}\boldsymbol{E})\right) \oplus (\oplus_{|\alpha|=s} \boldsymbol{W_1}(\boldsymbol{W}^{\propto}\boldsymbol{H}))$$
$$= (\oplus_{\substack{\mathbf{0}<|\alpha|<s \\ \alpha_1=1}} \boldsymbol{W}^{\propto}\boldsymbol{E}) \oplus (\oplus_{\substack{|\alpha|=s+1 \\ \alpha_1=1}} \boldsymbol{W}^{\propto}\boldsymbol{H})$$

By substation above in eq.(2) , we obtains :$\boldsymbol{H} = \boldsymbol{E} \oplus (\oplus_{\substack{\mathbf{0}<|\alpha|<s+1 \\ \alpha_1=0}} \boldsymbol{W}^{\propto}\boldsymbol{E}) \oplus (\oplus_{\substack{|\alpha|=s+1 \\ \alpha_1=0}} \boldsymbol{W}^{\propto}\boldsymbol{H}) \oplus \left(\oplus_{\mathbf{0}<|\alpha|<s+1} \boldsymbol{W}^{\propto}\boldsymbol{E}\right) \oplus (\oplus_{\substack{|\alpha|=s+1 \\ \alpha_1=1}} \boldsymbol{W}^{\propto}\boldsymbol{H})$, then:$\boldsymbol{H} = \boldsymbol{E} \oplus \left(\oplus_{\mathbf{0}<|\alpha|<s+1} \boldsymbol{W}^{\propto}\boldsymbol{E}\right) \oplus (\oplus_{|\alpha|=s+1} \boldsymbol{W}^{\propto}\boldsymbol{H}) = \left(\oplus_{|\alpha|<s+1} \boldsymbol{W}^{\propto}\boldsymbol{E}\right) \oplus (\oplus_{|\alpha|=s+1} \boldsymbol{W}^{\propto}\boldsymbol{H})$

Lemma (3) : The system $\{\boldsymbol{\varphi}_{k,j}\}_{j=0}^{2^k-1}$ (k - fixed) is orthogonal block.

Prove : $\{\boldsymbol{\varphi}_{k,j}\}_{j=0}^{2^k-1} = \{\boldsymbol{W}^{\propto}\boldsymbol{\varphi}\}_{\propto \in \Omega} \rightarrow \boldsymbol{W}^{\propto}\boldsymbol{\varphi} \in \boldsymbol{W}^{\propto}\boldsymbol{H}$,Since , $\boldsymbol{W}^{\propto}\boldsymbol{H} \perp \boldsymbol{W}^{\beta}\boldsymbol{H}$, $\propto \neq \beta$, $|\alpha| = |\beta| = k$, Also , $\boldsymbol{W}^{\propto}\boldsymbol{\varphi} \in \boldsymbol{W}^{\propto}\boldsymbol{H}$, and , $\boldsymbol{W}^{\beta}\boldsymbol{\varphi} \in \boldsymbol{W}^{\beta}\boldsymbol{H}$.Then we have : $(\boldsymbol{W}^{\propto}\boldsymbol{\varphi} , \boldsymbol{W}^{\beta}\boldsymbol{\varphi}) = \mathbf{0}$, and , $\{\boldsymbol{W}^{\propto}\boldsymbol{\varphi}\}_{|\alpha|=k}$is orthogonal block .

Lemma (4) : For all $\propto , \beta \in \Omega$, we have : $(\boldsymbol{w}_{\propto} , \boldsymbol{\varphi}_{\beta}) = \begin{cases} (\boldsymbol{w}_{\propto} , \boldsymbol{\varphi}) & if \propto = \beta\gamma \\ 0 & o.w. \end{cases}$

Proof : Write the Fourier - Walsh series of the function $\boldsymbol{\varphi}$as :$\boldsymbol{\varphi} = \sum_{\gamma\in\Omega}(\boldsymbol{\varphi} , \boldsymbol{w}_{\gamma})\boldsymbol{w}_{\gamma}$

Also , we have:$\boldsymbol{\varphi}_{\beta} = \boldsymbol{W}^{\beta}\boldsymbol{\varphi} = \sum_{\gamma\in\Omega}(\boldsymbol{\varphi} , \boldsymbol{w}_{\gamma})\boldsymbol{W}^{\beta}\boldsymbol{w}_{\gamma} = \sum_{\gamma\in\Omega}(\boldsymbol{\varphi} , \boldsymbol{w}_{\gamma})\boldsymbol{w}_{\beta\gamma}$,On other hand :$\boldsymbol{\varphi}_{\beta} = \sum_{\gamma\in\Omega}(\boldsymbol{\varphi}_{\beta} , \boldsymbol{w}_{\propto})\boldsymbol{w}_{\beta\propto}$, $\propto = \beta\gamma$

The coefficients of the Fourier - Walsh series are uniqueness . Also , we have :If $\propto = \beta\gamma$ for some $\gamma \in \Omega$. then$(\boldsymbol{\varphi}_{\beta} , \boldsymbol{w}_{\propto}) = (\boldsymbol{\varphi} , \boldsymbol{w}_{\gamma})$.If $\propto$can not be expressed as $\beta\gamma \forall \gamma \in \Omega$, then $(\boldsymbol{\varphi}_{\beta} , \boldsymbol{w}_{\propto}) = 0$.

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III. BIORTHOGONAL SYSTEM FOR AFFINE SYSTEM OF WALSH-PALEY SYSTEM TYPE

Denote by $\{x_n\}_{n\geq 0}$ the sequence of Fourier – Walsh coefficients of the function $\boldsymbol{\varphi}$. In view of the normalization in definition(1), we have $x_0 = 0$ and $x_1 = 1$. let us construct a new numerical sequence $\{y_n\}_{n\geq 0}$ with the same normalization $y_0 = 0$ and $y_1 = 1$ as follows . For natural numbers n , let us replace the index :$x_n = x_\propto$ and $y_n = y_\propto$, so that , for the empty sequence $\propto$, we have $x_\propto = y_\propto = 1$. The other $y_\propto$ are determined successively from the recurrence relations :
$\sum_{v=0}^{k} x(\alpha_1,\ldots,\alpha_k)y(\alpha_{v+1},\ldots,\alpha_k) = 0\ k = 1,2,\ldots\ldots$ (6)

Theorem (1) : The system of functions $\{\Psi_{\boldsymbol{n}}\}_{n\geq 0}$ is biorthogonal conjugate to the system $\{\boldsymbol{\varphi_n}\}_{n\geq 0}$ of a function $\boldsymbol{\varphi}$.

Proof : Suppose that $\propto,\beta\in\Omega$ with $|\propto| = k$, $|\beta| = l$. Let us calculate : $(\Psi_\propto,\boldsymbol{\varphi_\beta}) = \sum_{v=0}^{k} y(\alpha_{v+1},\ldots,\alpha k\,(w\alpha 1,\ldots,\alpha v,\boldsymbol{\varphi}\beta 1,\ldots,\beta l) = 0$,By using lemma (4) , we have : $(w(\alpha_1,\ldots,\alpha_v),\boldsymbol{\varphi}(\beta_1,\ldots,\beta_l)) = \begin{cases}(\boldsymbol{w_\propto},\boldsymbol{\varphi}) & ,\text{ if } (\alpha_1,\ldots,\alpha_v) = \beta\gamma \\ 0 & o.w.\end{cases}$ .The equation $(\alpha_1,\ldots,\alpha v = \beta\gamma$ can be solved γonly on in the case for which : $l\leq v$ and $\alpha 1=\beta 1$, . . . , $\alpha l=\beta l$. Moreover , we have $|\gamma| = v - l$ and $\alpha_{l+1} = \gamma,\ldots,\alpha_v = \gamma_{v-1}$. Thus , omitting the obvious zero summands , we finally obtain :$(\Psi_\propto,\boldsymbol{\varphi_\beta}) = \sum_{v=l}^{k} y(\alpha_{v+1},\ldots,\alpha_k)(w(\alpha_{l+1},\ldots,\alpha_v),\boldsymbol{\varphi}) = \sum_{v=l}^{k} x(\alpha_{l+1},\ldots,\alpha v\, y\alpha v+1,\ldots,\alpha k$.By using the recurrence relations , with replacement of $\alpha 1,\ldots,\alpha k$ by $\alpha l+1,\ldots,\alpha k$, we find that

$$\begin{cases}(\Psi_\propto,\boldsymbol{\varphi_\beta}) = 0 & , \quad for\ k\neq l \\ (\Psi_\propto,\boldsymbol{\varphi_\beta}) \neq 0 & , \ if \quad k = l\ ,if \quad \propto=\beta\end{cases}$$

Moreover $(\Psi_\propto,\boldsymbol{\varphi_\beta}) = x_1 y_1 = 1$.

Theorem (2) : Let be $\boldsymbol{\varphi}\in L^2(0,1)$, supp$\boldsymbol{\varphi}\subset[0,\ 1]$, $\int_0^1 \boldsymbol{\varphi(t)dt} = 0$. If the inequality $\sum_{k=0}^{\infty}(\sum_{j=0}^{2^k-1}|\boldsymbol{\varphi},\boldsymbol{w_{kj}}|^2)^{1/2} =$ c $< \infty$, then the affine system of Walsh type $\{\boldsymbol{\varphi_n}\}_{n\geq 0}$is Bessel system with Bessel constant B = max $\{1,c\}^2$.

Proof :$\boldsymbol{\varphi} = \sum_{\propto\in\Omega}\boldsymbol{x_\propto w_\propto}$- Fourier-Walsh series of function $\boldsymbol{\varphi}$.and,$\boldsymbol{p} = \sum_{\beta\in\Omega}\boldsymbol{c_\beta\varphi_\beta}$- polynomial of affine system $\{\boldsymbol{\varphi_n}\}_{n\geq 1}$ finite sum .
We consider for $k = 0,1,\ldots$,
$p_k = \sum_{|\propto|=k}\boldsymbol{x_\propto}\sum_{\beta\in\Omega}\boldsymbol{c_\beta w_{\beta\propto}}$ - Walsh -Paley polynomials , $\{\boldsymbol{w_{\beta\propto}} : |\propto| = k\ (k - fixed\ ),\ \boldsymbol{\beta}\in\Omega\}$ - orthogonal system $\boldsymbol{w_{\beta\propto}} = \boldsymbol{w_{\beta'\propto'}}$, $|\propto'| = k$, $\boldsymbol{\beta'}\propto'\in\Omega$, $\propto=\propto'$, $\boldsymbol{\beta} = \boldsymbol{\beta'}$.If $\boldsymbol{\beta}\propto=\boldsymbol{\beta'}\propto'$, then$|\propto| + |\boldsymbol{\beta}| = |\propto'| + |\boldsymbol{\beta'}|$, $|\propto| = |\propto'|$ and $|\boldsymbol{\beta}| = |\boldsymbol{\beta'}|$, $\propto=\propto'$and $\boldsymbol{\beta}=\boldsymbol{\beta'}$
$\|p_k\| = (\sum_{\substack{|\propto|=k\\ \beta\in\Omega}}|\boldsymbol{x_\propto c_\beta}|^2)^{1/2} = (\sum_{|\propto|=k}|\boldsymbol{x_\propto}|^2)^{1/2}(\sum_{\beta\in\Omega}|\boldsymbol{c_\beta}|^2)^{1/2}$,and ,$\sum_{k=0}^{\infty}\|p_k\| = (\sum_{\beta\in\Omega}|\boldsymbol{c_\beta}|^2)^{1/2}$. $\sum_{k=0}^{\infty}(\sum_{|\propto|=k}|\boldsymbol{x_\propto}|^2)^{1/2} < \infty$
wecalculate : $(p,\boldsymbol{w_\propto}) = \sum_{\beta\in\Omega}\boldsymbol{c_\beta}(\boldsymbol{\varphi_p},\boldsymbol{w_\propto}) = \sum_{\substack{\propto,\beta\\ \gamma=\beta\propto}}\boldsymbol{c_\beta}(\boldsymbol{\varphi},\boldsymbol{w_\propto})$ (by using Lemma (1)) :$(p,\boldsymbol{w_\propto}) = \sum_{\substack{\propto,\beta\\ \gamma=\beta\propto}}\boldsymbol{x_\propto c_\beta}$
$(\sum_{k=0}^{\infty} p_k\ ,\boldsymbol{w_\propto}) = \sum_{k=0}^{\infty}(p_k\ ,\boldsymbol{w_\propto}) = \sum_{k=0}^{\infty}\sum_{|\propto|=k} x_\propto\sum_{\beta\in\Omega}\boldsymbol{c_\beta}(\boldsymbol{w_{\beta\propto}},\boldsymbol{w_\propto}) = \sum_{\substack{\propto,\beta\\ \gamma=\beta\propto}}\boldsymbol{x_\propto c_\beta}$.This means :$\boldsymbol{p} = \sum_{k=0}^{\infty} p_k$!

$\|p\| \leq \sum_{k=0}^{\infty}\|p_k\| = \sum_{k=0}^{\infty}(\sum_{|\propto|=k}|\boldsymbol{x_\propto}|^2)^{1/2}(\sum_{\beta\in\Omega}|\boldsymbol{c_\beta}|^2)^{1/2}$, then we have :$\|\sum_{\beta\in\Omega}\boldsymbol{c_\beta\varphi_\beta}\| \leq \|\boldsymbol{\varphi}\|^*(\sum_{\beta\in\Omega}|\boldsymbol{c_\beta}|^2)^{1/2}$
It is equivalent to : $(\sum_{\beta\in\Omega}|f,\boldsymbol{\varphi_\beta}|^2)^{1/2} \leq \|\boldsymbol{\varphi}\|^*\|f\|$ - Bessel inequality .finally:$(\sum_{k=0}^{\infty}|(f,\boldsymbol{\varphi_n}|^2)^{1/2} \leq ((\int_0^1 \boldsymbol{f(t)\ dt})^2 + \sum_{\beta\in\Omega}|f,\boldsymbol{\varphi_\beta}|^2)^{1/2} \leq \max\{1,\|\boldsymbol{\varphi}\|^*\}.\|f\|$.

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IV. CONCLUSION

In our work , we give anew represention of functions in $L^2(0,1)$ by using affine system of Walsh-Paley system type . This representation in binary form which is can be use in communication system , signal processing , image processing , and others.

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