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ON AUTOMORPHISM OF LABELED SIMPLE CONNECTED GRAPH FROM PRESCRIBED DEGREES

Bidipta Majumder[*1], Subhajit Guha[2] and Soumyasri Ghosh[3]
[1,2,3]Department of Computer Science,
Barrackpore Rastraguru Surendranath College,
Barrackpore, West Bengal, India
bidipta2k8@gmail.com[1], guha.subhajit@gmail.com[2], soumyasri.ghosh@gmail.com[3]

Abstract: On counting of labeled connected graphs, the question comes into mind first; "How many ways a graph can be labeled?" As there exist certain number of labeled isomorphic graphs. On finding non isomorphic labeled graphs is an interesting problem itself. To provide the fact, graph automorphism should be considered.

Keywords: Degree sequence, graphic sequence, permutation, coset, orbit, stabilizer, isomorphism, automorphism

INTRODUCTION

A study of graphs as geometric objects necessarily involves the study of their symmetries, described by the group of automorphism. Indeed, there has been significant interaction between abstract group theory and the theory of graph automorphism, "leading to the construction of graphs with remarkable properties as well as to a better understanding and occasionally a construction or proof of nonexistence of certain finite simple groups"[8].

On the other hand, in contrast to classical geometries, most finite graphs have no automorphism other than the identity (asymmetric graphs), a fact that is largely and somewhat paradoxically responsible for its seeming opposite: "every (finite) group is isomorphic to the automorphism group of a (finite) graph"[8].

1. Literature Review

Different methods exist to prove a degree sequence to be graphic. In 1991, Sierksma and Hoogeveen [6] mentioned seven criteria for a sequence of integers to be graphic. Apart from these criteria Havel[18] and Hakimi[14][15] independently provide an algorithmic approach that allows for constructing a simple graph with a given degree sequence. Before, in 1960 Erdős–Gallai [16] gives a necessary and sufficient condition for a finite sequence of natural numbers to be the degree sequence of a simple graph; a sequence obeying these conditions is called "graphic". Isomorphism and Automorphism are the key terms in Group theory as well as in graph theory. In 1984 B.D. McKay and N.C. Wormald[9] wrote on Automorphism of random graphs with specified vertices. Famous mathematician László Babai[4] discussed on graph isomorphism and Automorphism property of two isomorphic graphs in the year 1994.

2. Integer Sequence of a Graph

Definition 1: A sequence $\zeta = d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers is called a degree sequence of given graph G, if the vertices of G can be labeled $v_1, v_2, v_3, \ldots v_n$ so that degree of $v_i = d_i$; for all i = 1, 2, 3,… n.

Definition 2: A sequence $\zeta = d_1, d_2, d_3, \ldots, d_n$ of nonnegative integers is said to be graphic sequence, if there exist a graph G whose degree d_i and G is called f realization of ζ. Any graphic sequence must satisfy the following conditions:

1. $d_i \le n-1$
2. $\sum_{i=1}^{n} di$ is even. However, these two conditions together do not ensure that a sequence will be graphic; for example the sequence 3, 3, 3, 1 is not graphic.

Theorem 1 If $d_1, d_2, d_3, \ldots, d_n$ is a non-negative integer sequence, then n specify the number of vertices $\frac{1}{2}\sum_{i=1}^{n} di$ specify the number of edges of a connected graph.

Lemma1 Let $d_1, d_2, d_3, \ldots, d_n$ be a non-negative graphic sequence and assume $d_j > d_k$. Then the sequence $d_1, \ldots. d_j - 1, \ldots d_k + 1, \ldots, d_n$ is also graphical.

2.1. Criteria for a Sequence of Integers to be Graphical

Theorem 2: (Havel [18] & Hakimi [14][15]) Consider the non-increasing sequence $S_1 = (d_1, d_2, \ldots, d_n)$ of nonnegative integers, where $n \ge 2$ and $d_1 \ge 1$. Then S_1 is graphical if and only if the sequence $S_2 = (d_2 - 1, d_3 - 1, \ldots, d_{d1+1} - 1, d_{d1+2}, \ldots, d_n)$ is graphical.

Proof Let the set of forbidden nodes be the empty set, X(i) = $\emptyset$. In this case L(i) ={1, 2, . . . , d_i} if i > d_i or L(i) = {1, 2, . . . , i − 1, i + 1, . . . , d_i + 1} when i ≤ di.Thus, according to the theorem, d is graphical if and only if $\{d_{1-1}, \ldots, d_{di-1}, d_{di+1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n\}$ is graphical when $i > d_i$, or $\{d_1 - 1, \ldots, d_{i-1} - 1, d_{i+1} - 1, \ldots, d_{di+1} - 1, d_{di+2}, \ldots, d_n\}$ is graphical when $i \le d_i$.

Theorem 3 (Erdős-Gallai[16]) Let $d = (d_1, d_2, \ldots, d_n)$ be a sequence of positive integers such that $d_i \ge d_{i+1}$. Then d is realized by a simple graph if and only if

i) $\sum_i di$ is even
ii) $\sum_{i=1}^{k} di \le k(k-1) + \sum_{i=k+1}^{n} \min\{k, di\}$ for all $1 \le k \le n-1$.

4. Groups and Graphs [10]

Definition 3. A non-empty set G is said to form a group with respect to a binary composition ∘ , if

i) G is closed under the composition ∘ , i.e., if a ∈ N and b ∈ N, then a ∘ b ∈ N .
ii) ∘ is associative, i.e., a ∘ (b ∘ c) = (a ∘ b) ∘ c
iii) There exists an element e in G such

that e ∘ a = a ∘ e = a, for all a in G.

iv) For each element a in G, there exists an element a’ in G such that a□ ∘ a = a ∘ a□ = e.

This Group is denoted by the symbol (G, ∘).

Definition 4 A permutation or symmetry of a set X is a 1-1 onto function σ: X → X. S_x denotes the set of all permutations of X; in the special case where X = {1, 2,… , n} we write S_n.

Example: S_4 has 4! = 24 elements. We may represent them in function table notation as

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \ etc.$$

Definition 5 The symmetric group S_n of degree n is the group of all permutations on n symbols. S_n is therefore a permutation group of order n! and contains as subgroups every group of order n.

Definition 6 Let S= {$a_1,a_2,\ldots\ldots,a_n$} a permutation f: S→S is said to be a cycle of length r, or an r-cycle if there are r elements $a_{i1},a_{i2},\ldots a_{ir}$ in S such that $f(a_{i1})= a_{i2}$, $f(a_{i2})= a_{i3},\ldots,f(a_{ir-1})= a_{ir}$, $f(a_{ir})= a_{i1}$.

The cycle is denoted by $(ai_1,ai_2,\ldots\ldots,a_{ir})$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \longrightarrow (1\ 2\ 3\ 4)$$

Definition 7 For H a subgroup of G and a ∈ G, the left coset of H by a is the set aH = {ah : h ∈ H}. Similarly the right coset of H by a is Ha = {ha : h ∈ H}.

Definition 8 If G acts on X and x ∈ X, then the stabilizer of x is Stab x = {g ∈ G : g(x) = x}, i.e., the set of permutations g in G which leave x unchanged, xg = x.

Definition 9 For x ∈ X, the orbit of x under G is Orb x = {g(x) : g ∈ G},i.e., the set of all images xg of x under all permutations g in G.

Theorem 3 (Orbit stabilizer theorem) If G acts on X and x ∈ X, then Stab x is a subgroup of G. If furthermore G is finite then

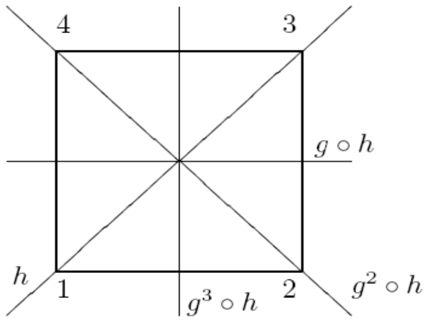
$$|G : \text{Stab } x| = |\text{Orb } x|.$$

Definition 10 Consider graphs G and H and a bijection f: V(G) → V(H). We call f an isomorphism from G to H when u, v∈E(G) iff, f (u)f (v)∈E(H). When such an f exists, we say that G is isomorphic to H and write G ≅ H.

Definition 11 An automorphism of a graph is an isomorphism from the graph to itself.

5. Finding Graph Automorphism

For the enumeration of labeled connected graphs, the question comes into mind first; ‘How many ways a graph can be labeled?’ As there exists certain number of labeled isomorphic graphs. On finding non isomorphic labeled graphs is an interesting problem itself. To provide an answer, we must consider the Automorphism (Aut) of a graph. Isomorphism of graphs is bijections of the vertex sets preserving adjacency as well as non-adjacency. Automorphism of the graph G = (V, E) are f: G → G isomorphism; they form the subgroup Aut(X) of the symmetric group Sym(V). The collection of all Automorphism of G, constitutes a group called the group of G. Thus the elements of Aut(G) are permutations acting on V. The orbit of a vertex v in a graph G is the set of all vertices α(v) such that α is an Automorphism of G.



e = (1)(2)(3)(4) g=(1234) g^2=(13)(24)
g^3= (4321) h= (1)(3)(24) g ∘ h= (14)(23)
g^2 ∘ h= (13)(2)(4) g^3 ∘ h= (12)(34)

Figure 1. Finding Automorphism number of the graph from the degree sequence 2,2,2,2

5.1. Some Elementary Facts About Automorphism Groups

Fact I: Let the components of X be $X_1,\ldots,X_k$. Then Aut(X) = $\prod_{i=1}^{k} \text{Aut}(Xi)$

This “fact” is not true except when the components are mutually non-isomorphic.

Fact II: For a simple graph X with edge-complement X□, we have Aut(X) = Aut(X□)

Theorem 4 [12]: The number of different ways in which the points of G can be labeled is n!/|Aut(G)|

Proof Since the theorem is obvious for p = 1, 2, we assume p >= 3. Now let G be the unlabeled graph on p points which corresponds to the function f mentioned above. It is clear that the number of ways in which G can be labeled is simply the number of functions in the orbit of f regarded as an element in the object set of the power group[13] . Furthermore, the stabilizer of f in is obviously isomorphic to r(G). Applying the lemma to this power group, we have the result that the number of ways of labeling G is the order of divided by the order of r(G), i.e., the index of r(G) regarded as a subgroup of the power group. The proof is completed by observing that the order of this power group is p! when p ≥ 3.

6. Algorithmic Outline

Input: An Integer sequence in non-increasing order of degrees.

Output: Automorphism numbers

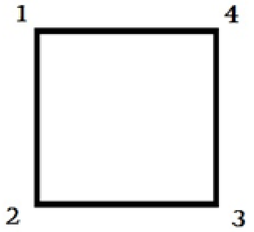
Step 1. Check whether the integer sequence produces at least one connected graph or not.

Step 2. Check whether the sequence is graphic sequence or not.

Step 3. For the given graphic sequence generate a simple connected graph G.

Step 4. Finding the Automorphism number (|Aut (G)|).

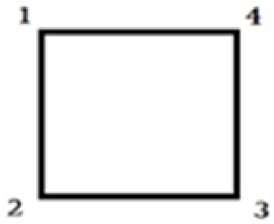
Let's take an example, for a given degree sequence 2,2,1,1 . The graph or adjacency matrix generated by D.E. Knuth's [1] tabular approach is



	1	2	3	4
1	0	1	0	1
2	1	0	1	0
3	0	1	0	1
4	1	0	1	0

Figure 2. Generation of the graph from degree sequence 2,2,2,2

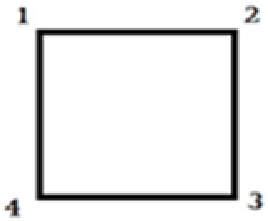
The automorphism of the above graph are



$\lambda_0 = (1)(2)(3)(4)$

	2	1	4	3
2	0	1	0	1
1	1	0	1	0
4	0	1	0	1
3	1	0	1	0

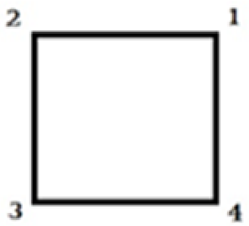
(a)



$\lambda_1 = (1)(3)(24)$

	4	1	2	3
4	0	1	0	1
1	1	0	1	0
2	0	1	0	1
3	1	0	1	0

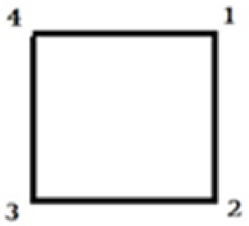
(b)



$\lambda_2 = (4321)$

	2	3	4	1
2	0	1	0	1
3	1	0	1	0
4	0	1	0	1
1	1	0	1	0

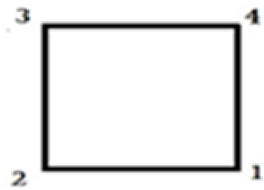
(c)



$\lambda_3 = (14)(23)$

	4	3	2	1
4	0	1	0	1
3	1	0	1	0
2	0	1	0	1
1	1	0	1	0

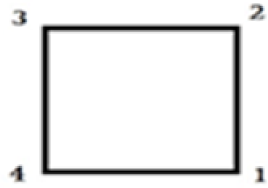
(d)



$\lambda_4 = (13)(2)(4)$

	3	2	1	4
3	0	1	0	1
2	1	0	1	0
1	0	1	0	1
4	1	0	1	0

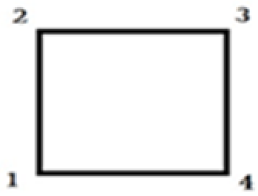
(e)



$\lambda_5 = (13)(24)$

	3	4	1	2
3	0	1	0	1
4	1	0	1	0
1	0	1	0	1
2	1	0	1	0

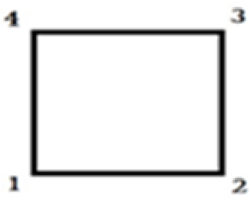
(f)



$\lambda_6 = (12)(34)$

	1	2	3	4
1	0	1	0	1
2	1	0	1	0
3	0	1	0	1
4	1	0	1	0

(g)



$\lambda_7 = (1234)$

	1	4	3	2
1	0	1	0	1
4	1	0	1	0
3	0	1	0	1
2	1	0	1	0

(h)

Figure 3.(a-h).Automorphic graphs of Figure 2

7. Case Studies

There exist some degree sequences from where more than one non-isomorphic unlabeled graphs are present. The following graphs (from Figure 4.to Figure 7.) are the exceptions where more than one non-isomorphic unlabeled graphs from a prescribed degree sequence is present.

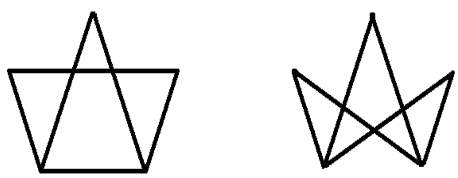


Figure 4. For degree sequence 3,3,2,2,2 with vertex 5.

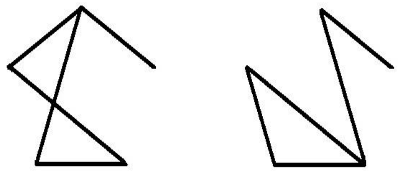


Figure 5. For degree sequence 3,2,2,2,1 with vertex 5.

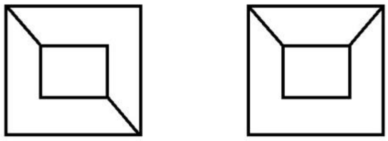


Figure 6. For degree sequence 3,3,3,3,2,2,2,2 with vertex 8.

Figure 7. For degree sequence 5,5,5,3,3,3,2,2,2 with vertex 9.

8. RESULTS

Table 1. Automorphism number of the graphs from prescribed degree sequences

No. of vertex	Degree sequence	Automorphism number
3	1,1,1	not graphic
	2,1,1	2
	2,2,2	6
4	1,1,1,1	Disconnected
	2,1,1,1	not graphic
	2,2,1,1	2
	3,1,1,1	6
	3,2,2,1	2
	3,3,2,2	4
5	2,2,2,2,2	10
	3,3,2,2,2	2
	4,4,2,2,2	12
	4,4,4,4,4	10
6	4,4,2,2,2,2	4
	3,3,3,2,2,2	not graphic
	3,3,3,2,2,1	1
	3,3,3,1,1,1	6
7	2,2,2,2,2,1,1	2
	2,2,2,2,2,2,2	14
	3,2,2,2,1,1,1	1
8	3,2,2,2,2,1,1,1	2
	3,2,2,2,2,2,2,1	2
	3,3,3,1,1,1,1,1	8
9	5,5,5,3,3,3,2,2,2	1

8. Conclusion

Graph automorphism is an interesting problem. Different unlabeled connected graphs with same graphic sequence are present, for labeled graph enumeration it seems ambiguous. Hence, graph automorphism becomes a major part for unlabeled graph generation.

9. Acknowledgement

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