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# Lepton Residual Symmetries

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## Research Article

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**ABSTRACT**

Some aspects concerning the flavor symmetry were studied. Two approaches are discussed to deal with flavor symmetry. The desired lepton mixing matrix determines the flavor symmetry group up to a certain representation.

The dependence of the lepton mixing matrix on the representation of the symmetry group accounts for the differences between models used the same symmetry group.

We discuss the connection between the two approaches to fix flavor assignments and alignments of flavor vacuum.

Another approach is called bottom- up or the residual symmetry approach, at which the lepton mass matrices are considered to have remnant symmetry of breaking discrete group. The goal of this approach is to get this discrete group.

## INTRODUCTION

Many aspects such as the differences in mixing and mass hierarchy for lepton and quark sectors force the flavor symmetry to be proposed to account for these aspects. Several models based on discrete symmetries were proposed to account for flavor aspects [1-7]. For most of these models, some additional Guage scalars (flavons) were considered beside a lot of assumptions and extra symmetries were proposed to account for experimental data. This is called up-bottom approach in which the Lagrangian is considered to be invariant under a discrete group and each flavor is assigned to one of the irreducible representations of the discrete group. After spontaneous symmetry breaking, the recent data of neutrino masses and mixing have to be recovered. Another approach is called bottom- up or the residual symmetry approach, at which the lepton mass matrices are considered to have remnant symmetry of breaking discrete group. The goal of this approach is to get this discrete group. In this paper, we try to summarize the relations between these two approaches and study some aspects of the flavor symmetries.

## LEPTON RESIDUAL SYMMETRIES

Let the Lagrangian of the lepton sector be invariant under a horizontal symmetry group G that is spontaneously broken allowing the flavors to acquire their masses. The group G is not broken completely, some of its elements remain unbroken and keep the mass matrices of neutrinos and charged leptons invariant. We refer to the unbroken symmetry manifested in the mass matrices as the residual symmetry. Let Mv be the mass matrix of the effective light Majorana neutrinos. This mass matrix can be diagonalized by a unitary matrix Uv

$$U_{\nu}^{T} M_{\nu} U_{\nu} = M_{\nu}^{Diag} \tag{1}$$

where $M_{\nu}^{Diag}$ = diag(m1, m2, m3). If the $M_{\nu}$ manifests such symmetry, one can find unitary matrices Si (the unbroken elements of G) that keep the matrix $M_{\nu}$ invariant,

$$S_{i}^{T} M_{\nu} S_{i} = M_{\nu} \tag{2}$$

If the matrices $S_i$ are real, $M_{\nu}$ and $S_i$ commute. In the case of non-degenerate neutrino mass spectrum, $M_{\nu}$ and $S_i$ can be diagonalized simultaneously by $U_{\nu}$,

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$$U_\nu^\dagger S_i U_V = S_i^d \tag{3}$$

where $S_i^d$ is a diagonal matrix containing the eigenvvalues of $S_i$ . From Eqs. (1), (2)

$$U_\nu^t S_i^t U_\nu^* U_\nu^t M_\nu U_\nu U_\nu^\dagger S_i U_\nu = M_\nu^{Diag} \tag{4}$$

$$S_i^{d^t} M_\nu^{Diag} S_i^d = M_\nu^{Diag}$$

$$S_i^{d^t} S_i^d = S_\nu^{d^2} \tag{5}$$

Therefore, the eigenvalues of the symmetry matrix Si are ± 1. The three possible diagonal matrices of Si-up to sign- are

$$S_1^d = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}, S_2^d = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, S_3^d = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{6}$$

From Eq (3) the symmetry matrices Si are just unitary transformations of the above three matrices $S_{di}$, i=1, 2, 3. The symmetry matrices Si can be calculated by

$$S_i = U_\nu S_i^d U_i^\dagger$$
$$U_l m_l V_l = m_l^{Diag} \tag{7}$$

Regardless the form of the mass matrix Mv, it has Z2 × Z2 residual symmetry [8], provided that it has three distinct eigen values [9]. We can apply the same procedures for charged leptons. In general, the charged lepton mass matrix ml is not symmetric nor Hermitian, so it can diagonalized by two unitary matrices,

$$U_l m_l V_l = m_l^{Diag} \tag{8}$$

where $m_l^{Diag}$ = diag($m_e$, $m_\mu$, $m_\tau$ ). It is convenient to use the Hermitian matrix Ml = $m_l m_l$ to deal only with the left handed mixing $U_l$ which contributes in lepton mixing $U_{PMNS}$. It can be diagonalized by $U_l$ ,

$$U_l^\dagger M_l U_l \;=\; \left(m_l^{Diag}\right)^2 \tag{9}$$

We can find unitary matrices Ti that keep the $M_l$ invariant:

$$T_\alpha^\dagger M_l T_\alpha = M_l \;,\; \alpha = e, \mu, \tau \tag{10}$$

The matrix $T_\alpha$ and $M_l$ commute, and can be diagonalized simultaneously by $U_l$ ,

$$T_\alpha \;=\; U_l T_\alpha^d U^\dagger \tag{11}$$

It is easy to prove that $T_\alpha^{d^\dagger} T_\alpha^d = 1$ = 1. Any diagonal unitary matrix can be a solution of $T_i^d$ . If the matrices Ul are orthogonal rather than unitary, the mass matrices Ml are symmetric, the only solutions- up to sign- are the three matrices $S_i^d$ in Eq 6. Without loss of generality, we can choose the three 4 matrices [10]

$$T_e^d = diag\left\{1,\; e^{2\pi ik/m,\; e-2\pi ik/m}\right\} \tag{12}$$

$$T_\mu^d = diag\left\{e^{2\pi ik/m,\; 1,\; e-2\pi ik/m}\right\} \tag{13}$$

$$T_\tau^d = diag\left\{e^{2\pi ik/m,\; e-2\pi ik/m},\; 1\right\} \tag{14}$$

where m must satisfy $\left(T_\alpha^d\right)^m = I$ . and k is an integer. As we see, according to the forms of the mixing matrices $U_\nu$ and $U_l$ we can find from Eqs (7) and (11) the unbroken elements of the flavor group G which are responsible of the residual symmetry in neutrino and charged lepton mass matrices. In the so called tri-bimaximal symmetry (TBM), $\text{Sin}^2\,\theta_{12} = \frac{1}{3}$, $\text{Sin}^2\,\theta_{23} = \frac{1}{2}$, $\text{Sin}\theta_{13} = 0$. It is considered due to the appearing symmetry in the neutrino mass matrix,

$$M_\nu = \begin{pmatrix} a & b & c \\ \ldots & \frac{1}{2}(a+b+c) & \frac{1}{2}(a+b-c) \\ \ldots & \ldots & \frac{1}{2}(a+b+c) \end{pmatrix}$$

In the case of TBM mixing, in the bases at which the charged lepton mass matrix is diagonal, and hence the total lepton mixing comes from the neutrino sector, the three symmetry matrices for neutrino are

$$S_1=\frac{1}{3}\begin{pmatrix}1&-2&-2\\-2&-2&1\\-2&1&-2\end{pmatrix},S_2=\frac{1}{3}\begin{pmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{pmatrix},S_3=\frac{1}{3}\begin{pmatrix}1&0&0\\0&0&-1\\0&-1&0\end{pmatrix} \tag{15}$$

The horizontal symmetry group G is the group generated by the matrices $\{S_i, T_\alpha\}$. One can choose one of the $S_i$ and one of $T_\alpha$ and consider the other matrices of $S_i$ and $T_\alpha$ as accidental symmetries of the mass matrices 5 of neutrino and charged leptons [10]. It was claimed by using all Si matrices and the Te as the generators of the horizontal symmetry group, that $S_4$ is the unique symmetry of tri-bimaximal in lepton sector and any group can explain the leptonic horizontal symmetry must contain $S_4$ as subgroup [11-13]. It is worth to notice that: The horizontal flavor group depends on the leptonic mixing matrix, i.e. if the mixing matrix is changed, the residual symmetry in neutrino sector and that in charged lepton sector remain the same but in new representation while the group G that generated by the $S_i$ and $T_\alpha$ matrices will be changed. Let us illustrate these results in details, if the horizontal group G generated by matrices $S_i$ and $T_\alpha$- the residual symmetry matrices in neutrino and charged lepton sectors respectively- is arisen by a leptonic mixing matrix U, we refer to this as old case. On the other hand, if G' the horizontal group generated by $S_\alpha^0$ and $T_\alpha^0$ is arisen by a mixing matrix $U_0$, we refer to this as the new case. If the mass eigenvalues are the same in both cases, and the mixing matrices in each sector are related to each other by a unitary rotation as the following

$$U_\nu^0 = W_1 U_\nu,\; U_l^0 = W_2 U_l \tag{16}$$

For neutrino sector, the mass matrices in both cases are related as the following

$$\begin{aligned} M_\nu^0 &= U_\nu^0 M_\nu^{Diag}\ U_0\ T_\nu \\ &= W_1 U_\nu M_\nu^{Diag}\ U_\nu^T\ W_1^T \\ &= W_1 M_\nu W_1^T \end{aligned} \tag{17}$$

The residual symmetry matrices in both cases are related as:

$$\begin{aligned} M_\nu &= S_T M_\nu S = S^T W_1^\dagger M_\nu' W_1^* S \\ M_\nu' &= W_1 M_\nu W_1^T = W_1 S^T W_1^\dagger M_\nu' W_1^\dagger S W_1^T \\ M_\nu' &= S'^T M_\nu'\ S' \end{aligned} \tag{18}$$

So

$$S' = W_1^* S W_1^T \tag{19}$$

For charged lepton sector, the mass matrices in both cases are related as the following

$$\begin{aligned} M_l' &= U_l' \left(m_l^{diag}\right)^2 U_l'^\dagger \\ &= W_2 U_l \left(m_l^{diag}\right)^2 U_l^\dagger\ \mathrm{W}_2^\dagger \\ &= W_2 M_l \mathrm{W}_2^\dagger \end{aligned} \tag{20}$$

The residual symmetry matrices in both cases are related as:

$$\begin{aligned} M_l &= T^\dagger M_l T = T^\dagger W_2^\dagger M_l' W_2 T \\ M_l' &= W_2 M_\nu W_2^\dagger = W_2 T^\dagger W_2^\dagger M_l' W_2 T W_2^\dagger \\ M_l' &= T\ 0\ \dagger M_l'\ T' \end{aligned} \tag{21}$$

So

$$T' = W_2 T W_2'^\dagger \tag{22}$$

From Eqs (19), (22) we find that the residual symmetry matrices in the new case are related by unitary similarity transformations to the old residual 7 symmetry matrices, so the residual symmetries remain the same but in new representation, but not all the generators of G are transformed by the same similarity transformations. So the symmetry group G depends on leptonic mixing matrix U. The new leptonic mixing is related to the old one by

$$U' = U_l'^\dagger U_\mathrm{v}' = U_l^\dagger W_2^\dagger W_1 U_\nu \tag{23}$$

If W1 = W2 the mixing matrices are the same $U'$=U but the bases in the two cases are different, Si and Tα transform by the same similarity transformation, the horizontal group G remain unchanged but in different representation. So we can say that the same mixing matrix U can lead to the same the same group G but in different representation depending on the bases of representation of the fields and mass matrices

## CONNECTION BETWEEN RESIDUAL SYMMETRY AND UP-BOTTOM

In general, consider generic Yukawa coupling term

$$L = h_{ijk}\psi_i\varphi_j\theta_k + h.c. \tag{24}$$

Assume that the Lagrangian L is invariant under a horizontal symmetry group G and ψ and φ transform according to the irreducible representations (IR) $\Gamma^\alpha$ and $\Gamma^\beta$ of G, respectively, the IR $\Gamma_\gamma$ is a result of the tensor product $\Gamma^\alpha * \Gamma^\beta$ and θ transforms according to the complex-conjugate IR $\Gamma^{\gamma *}$ [13] .

$$\psi_a^\alpha \rightarrow \Gamma_{am}^\alpha(T)\psi_m^\alpha,\ \varphi_b^\beta \rightarrow \Gamma_{bk}^\beta(T)\varphi_k^\beta\ ,\ \theta_c^\gamma \rightarrow \Gamma_{ci}^\gamma(T)\theta_i^\gamma \tag{25}$$

for every T ∈ G.

Consider (αa, βb|γci) be the Clebsch-Gordan (CG) coefficient of the group G, mixing the states $\psi_a^\alpha$ in IR $\Gamma^\alpha$ with the orthonormal states $\varphi_b^\beta$ transforming as IR $\Gamma^\beta$. The results are an orthonormal state |θ γ c i in IR $\Gamma^\gamma$

$$\begin{aligned}&\left[\theta_c^\gamma\right) = \sum_{a,b}\left(\psi_a^\alpha\right]\left|\varphi_b^\beta\right\rangle\langle\alpha a,\ \beta b, \gamma c i\rangle,\\ &\left[\psi_a^\alpha\right)\left[\varphi_b^\beta\right) = \sum_{\gamma,c}\left(\theta_c^\gamma\right]\left|\gamma c\right\rangle\langle\alpha a,\ \beta b\rangle\end{aligned} \tag{26}$$

Mu ply from left the projection operator P(T) of the group G with the second line in Eq (26)

$$P(T)\left|\psi_a^\alpha\right\rangle\left|\varphi_b^\beta\right\rangle = \sum_{\gamma,c}P(T)\ \left|\theta_c^\gamma\right\rangle\left\langle\gamma c\right|\alpha a,\ \beta b i\rangle$$

$$\left(P(T)\left|\psi_a^\alpha\right\rangle\right)\ \left(P(T)|\left|\varphi_b^\beta\right\rangle\right)\ = \sum_{\gamma,c}P(T)\ \left|\theta_c^\gamma\right\rangle\left\langle\gamma c\right|\alpha a,\ \beta b i\rangle$$

Multiply from left with

$$\left|\theta_c^\gamma\right\rangle = \sum_{a',b'}\left\langle\gamma c'\right|\alpha a',\ \beta b'\rangle\left\langle\psi_{a'}^\alpha\right|\left\langle\varphi_{b'}^\beta\right|$$

$$\sum_{a',b'}\left\langle\gamma c'\right|\alpha a',\ \beta b'\rangle\left\langle\psi_{a'}^\alpha\right|\left\langle\varphi_{b'}^\beta\right|P(T)\left|\psi_a^\alpha\right\rangle\left|\varphi_b^\beta\right\rangle\left(P(T)|\left|\varphi_b^\beta\right\rangle\right) = \sum_c P(T)\left|\theta_{c'}^\gamma\right\rangle P(T)\ \left|\theta_c^\gamma\right\rangle\left\langle\gamma c\right|\alpha a,\ \beta b i\rangle$$

$$\sum_{a',b'}\left\langle\gamma c'\right|\alpha a',\ \beta b'\rangle\Gamma_{a'a}^\alpha(T)\Gamma_{b'b}^\beta(T) = \Gamma_{c'c}^\gamma(T)\left\langle\gamma c\right|\alpha a,\ \beta b i\rangle \tag{27}$$

By the same way,

$$\sum_{a',b'}\Gamma_{a'a}^\alpha(T)\Gamma_{b'b}^\beta(T)\left\langle\gamma c'\right|\alpha a',\ \beta b'\rangle = \sum_c\left\langle\gamma c\right|\alpha a,\ \beta b i\rangle\Gamma_{c'c}^\gamma(T) \tag{28}$$

From Eqs (24), (26), the Yukawa coupling hijk is proportional to the Clebsch-Gordan coefficient,

$$hijk = hh\left\langle\alpha * i,\ \beta j\right|\gamma k\rangle \tag{29}$$

where h is a free parameter.

Now suppose that the neutrino masses arise from the type-I seesaw mechanism, then the Lagrangian responsible for flavor structure of mass matrices can be written as

$$L = -y_\gamma\tilde{L}\ He_R\ \varphi^\gamma - h_\gamma\bar{L}H\ \tilde{\nu}_R\xi^\gamma - h'_\gamma\nu_R^c\ \nu_R\ \chi^\gamma + h.c., \tag{30}$$

where L is the SU(2) lepton doublet, $e_R$ is the right-handed charged leptons, $\nu_R$ is the right handed neutrinos, and φ, ξ and χ are flavons which are Gauge singlets. Every term is assumed to be invariant under a horizontal symmetry group G, with y, h, h' being the Yukawa coupling constants. Take the charged lepton term as an example, suppose that L and eR transform under IRs $\Gamma^\alpha$ and $\Gamma^\beta$ respectively and the flavon is transformed under an IR $\Gamma^\gamma$ that occurs in the tensor product $\Gamma^\alpha * \Gamma^\beta$, therefore we sum over all IR $\Gamma^\gamma$ appear in this tensor product. After spontaneous symmetry breaking the flavon φ acquire a vacuum expectation value to get the mass of charged leptons. From Eqs 24, 29, we can write the Yukawa coupling term of the charged leptons explicitly,

$$L_e = -\sum_{a,b}\left[\bar{L}\right](e_R)_b\sum_{c,\gamma}y_\gamma\left\langle\alpha * a,\ \beta b\right|\gamma c\rangle\left\langle\varphi_c^\gamma\right| \tag{31}$$

Then the charged lepton mass is

$$\left(m_e\right)_{ab} = \sum_{c,\gamma}y_\gamma\left\langle\alpha * a,\ \beta b\right|\gamma c\rangle\left\langle\varphi_c^\gamma\right\rangle \tag{32}$$

The horizontal symmetry group G is broken spontaneously via the vacuum expectation values of the flavons φ, χ and χ'. The symmetry group G is broken to a smaller symmetry group G', we call this smaller symmetry as residual symmetry. The generators of the symmetry group G' are the elements of G that keep the vacuum expectation values of the flavons invariant. If $F_i$ are the elements of G that keep the vacuum expectation value of φ invariant, then

$$\Gamma^{\gamma}\left(F_i\right)\left\langle\varphi^{\gamma}\right\rangle=\left\langle\varphi^{\gamma}\right\rangle, \tag{33}$$

where i = 1, 2, 3. From Eqs. (28), (33),

$$\left(m_e\right)_{ab}=\sum_{c,\gamma} y_\gamma\left\langle\alpha * a,\ \beta b\middle|\gamma c\right\rangle\left\langle\varphi_c^{\gamma}\right\rangle=\sum_{c,\gamma} y_\gamma\left\langle\alpha * a,\ \beta b\middle|\gamma c\right\rangle\left\langle\varphi_c^{\gamma}\right\rangle\Gamma_{cc'}^{\gamma}(F_i)\left\langle\varphi_{c'}^{\gamma}\right\rangle \tag{34}$$

$$\sum_{\gamma,a',b} y_\gamma\,\Gamma_{aa'}^{\alpha\dagger}(F_i)\Gamma_{bb'}^{\beta}(F_i)\left\langle\alpha * a',\ \beta b'\middle|\gamma c'\right\rangle\left\langle\varphi_c^{\gamma}\right\rangle \tag{35}$$

Consequently, we end to the usual form of the mass invariance equation,

$$\Gamma^{\alpha\dagger}\left(F_i\right)\ m_e\ \Gamma^{\beta}\left(F_i\right)=m_e \tag{36}$$

Similarly, for the neutrino sector, if Si are the elements of G that keep the vacuum expectation value of ξ and χ invariant, then

$$\Gamma^{\gamma}\left(S_i\right)h\left\langle\xi^{\gamma}\right\rangle=\left\langle\xi^{\gamma}\right\rangle,\quad \Gamma^{\gamma}\left(Si\right)\left\langle\chi^{\gamma}\right\rangle=\left\langle\chi^{\gamma}\right\rangle \tag{37}$$

If $v_R$ transforms under IR $\Gamma^{\beta}$, the Dirac neutrino mass matrix and right handed neutrino mass are

$$\left(m_D\right)_{ab}=\sum_{c,\gamma} h_\gamma\ \left\langle\alpha a,\ \beta b\middle|\gamma c\right\rangle\left\langle\xi_c^{\gamma}\right\rangle,\left(m_R\right)_{ab}=\sum_{c,\gamma} h'_\gamma\ \left\langle\beta a,\ \beta b\middle|\gamma c\right\rangle\left\langle\chi_c^{\gamma}\right\rangle \tag{38}$$

Similar to the case of charged leptons Eq (36), the mass matrix invariance equations in neutrino sector can be of the form:

$$\Gamma^{\alpha\dagger}\left(S_i\right)\ m_D\Gamma^{\beta}\left(S_i\right)=m_D,$$
$$\Gamma^{\beta\dagger}\left(S_i\right)M_R\ \Gamma^{\beta *}\left(S_i\right)=M_R,$$
$$\Gamma^{\alpha^T}\left(S_i\right)\ M_\nu\ \Gamma^{\alpha}\left(S_i\right)=M_\nu, \tag{39}$$

Where

$$M_\nu=m_D^T\ M_R^{-1} m_D \tag{40}$$

## CONCLUSION

The previous discussion we link between the generators of the residual symmetry ($S_i$, $F_i$) and the dynamical inputs (IRs of fields and vacuum alignments). If we consider a certain group as horizontal symmetry group such as A4 or S4, we can use Eqs (39) to write constrains on the Irreducible representations that lead to the desired mixings. The group A4 is generated by T d e in Eq (12) with k=1, m=3 and S2 in Eq (15), so the vacuum alignments of the flavons φ and χ that are kept invariant under the action of $T_{de}$ and S2 are

$$\left\langle\varphi\right\rangle\ =(1,0,0),\left\langle X\right\rangle=(1,1,1)$$

For the group S4, it is generated by the matrices $T_{de}$ , S1, S2 and S3 in Eqs (12, 15) with k=1, m=3, so the vacuum alignments of the flavons are as follows,

In the irreducible representation 2

$$\left\langle\varphi\right\rangle\ =(0,0),\left\langle X\right\rangle=(1,1)$$

In the irreducible representation 3,

$$\left\langle\varphi\right\rangle\ =(1,0,0),\left\langle X\right\rangle=(0,0,0)$$

In the irreducible representation 3',

$$\left\langle\varphi\right\rangle\ =(1,0,0),\left\langle X\right\rangle=(1,1,1)$$

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