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Fixed Points of Different Contractive Type Mappings on Tensor Product Spaces

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ABSTRACT: In this paper, we derive some fixed point theorems in the projective tensor product $(X\otimes_\gamma Y)$ of two Banach spaces X and Y. Using two mappings $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$, we construct a self-mapping T on $X\otimes_\gamma Y$. Taking $T_1$and T_2 as different contractive type mappings, we study the characteristics of the mapping T and the existence and the uniqueness of the fixed point of T in the closed unit ball of $X\otimes_\gamma Y$. The converse of this result is also discussed here.

KEYWORDS: projective tensor product, contractive type mappings, asymptotically regular property

I. INTRODUCTION

Banach's contraction mapping principle [2] has been the source of metric fixed point theory with its wide applicability in different branches of mathematics. Kannan, in [7], developed a substantially new contractive mapping to prove the fixed point theorem. From this time, a number of researchers, viz., Boyd and Wong [3], Chatterjea [5] Ciric[6], Reich [11], Rhoades [12], Saha and Dey[8] have tried to prove the fixed point theorems with different approaches using more generalized contractive mappings. In 2011, Saha,Dey and Ganguly[9]discussed fixed point theorems for contraction mappings with asymptotically regularity for integral setting.

In this paper, we study some fixed point theorems in the projective tensor product of two Banach spaces. Let X and Y be two different Banachspaces and $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$ be two operators.Using $T_1$and T_2 we study the existence and uniqueness of fixed point of an operator T on the space$X\otimes_\gamma Y$.

Before discussing the main results, we first recall some basic definitions (refer to [9], [11]).

Definition 1.1 Let X and Y be normed spaces. A mapping $T: X \longrightarrow Y$ is called non-expansive if and only if $\|Tx - Ty\| \leq \|x - y\|\ \forall\, x, y \in X$.

A mapping $T: X \longrightarrow Y$ is called contraction if and only if $\|Tx - Ty\| \leq c\|x - y\|$, where c is real number with $0 \leq c < 1$ for all $x, y \in X$.

Clearly, contraction⇒ non expansive and all such mappings are continuous.

In 1966, Browder and Petryshyn [4] investigated the asymptotically regular property for a self-map on a metric space (X, d).

Definition 1.2A mapping $T: X \longrightarrow X$ is called asymptotically regular at a point $x \in X$ iff $\|T^{n+1}x - T^n x\| \to 0$ as $n \to \infty$, where $T^n x$ denotes the n-th iterate of T at $x \in X$.

It is proved by researchers that this property is necessary for some contractive mappings to produce fixed points.

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II. MAIN RESULTS

The following Lemma plays an important role in proving our main results.

Lemma 2.1 Let X and Y be Banach spaces and $X\otimes_\gamma Y$ be their projective tensor product. If $T: X\otimes_\gamma Y \longrightarrow X\otimes_\gamma Y$ is a contraction mapping then

(i)$T$is asymptotically regular at every $u = \sum_i x_i \otimes y_i \in X\otimes_\gamma Y$
(ii) For each $u = \sum_i x_i \otimes y_i \in X\otimes_\gamma Y$, the sequence $\{T^n(\sum_i x_i \otimes y_i)\}$ is convergent in $X\otimes_\gamma Y$, and every such sequence has the same limit.

Proof. (i) Since T is a contraction mapping, so, there exists some constant c with $0 \le c < 1$ such that $\|Tu - Tv\| \le c\|u - v\|$ for all $u = \sum_i x_i \otimes y_i$, $v = \sum_i x_i' \otimes y_i' \in X\otimes_\gamma Y$.
Now, $\|T^{n+1}u - T^n u\| = \|T(T^n u) - T(T^{n-1}u)\| \le c\|T^n u - T^{n-1}u\| \le \cdots \le c^n\|Tu - u\|$
$\to 0 \; as \; n \to \infty \qquad (\because c < 1)$

(ii) $\|T^{n+k}u - T^n u\| \le \|T^{n+k}u - T^{n+k-1}u\| + \|T^{n+k-1}u - T^{n+k-2}u\| + \cdots + \|T^{n+1}u - T^n u\|$
$\to 0 \; as \; n \to \infty$ (using (i))
Thus $\{T^n(\sum_i x_i \otimes y_i)\}$ is a Cauchy sequence in $X\otimes_\gamma Y$, and so, it converges to some $\alpha \in X\otimes_\gamma Y$.Let $\{T^n(\sum_i x_i' \otimes y_i')\}$ be another Cauchy sequence in $X\otimes_\gamma Y$ which converges to some $\beta \in X\otimes_\gamma Y$.

$$\|\alpha - \beta\| = \left\| \lim_{n\to\infty} T^n u - \lim_{n\to\infty} T^n v \right\| = \lim_{n\to\infty} \|T^n u - T^n v\| \le c^n \|u - v\| = 0$$

Thus $\alpha = \beta$ and so, the limit of all sequences $\{T^n(\sum_i x_i \otimes y_i)\}$ is same.□

Let $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$ be two contraction mappings. We define $T: X\otimes_\gamma Y \longrightarrow X\otimes_\gamma Y$ by $Tu = T_1 u \otimes T_2 u$, $u \in X\otimes_\gamma Y$. Let $B_{X\otimes_\gamma Y}$ denote the closed unit ball in $X\otimes_\gamma Y$.Now, we prove:

Theorem 2.2 If $0 \in$ Ker $T_1 \cap$ KerT_2, then in $B_{X\otimes_\gamma Y}$, T has a unique fixed point at 0.

Proof. For $u, v \in X\otimes_\gamma Y$, let $\|T_1 u - T_1 v\| \le c_1\|u - v\|$and $\|T_2 u - T_2 v\| \le c_2\|u - v\|$, where $0 \le c_1, c_2 < 1$
$\|Tu - Tv\| \le \|T_1 u \otimes T_2 u - T_1 v \otimes T_2 v\| \le \|(T_1 u - T_1 v)\otimes T_2 v\| + \|T_1 u \otimes (T_2 u - T_2 v)\|$
$\le c_1\|u - v\|\|T_2 v\| + \|T_1 u\| c_2 \|u - v\|$ ……… (2.1)
Since$T_1(0) = 0$, $\|T_1 u - T_1 0\| \le c_1\|u - 0\| \Rightarrow \|T_1 u\| \le c_1\|u\|$.Similarly,$\|T_2 u\| \le c_2\|u\|$
So, (2.1) $\Rightarrow \|Tu - Tv\| \le c_1\|u - v\| c_2 \|v\| + c_1\|u\| c_2 \|u - v\|$
$\le c_1 c_2 \|u - v\|(\|v\| + \|u\|) \le c_1 c_2 (\|u\| + \|v\|)^2$ ……… (2.2)
Taking $v = 0$ we have $T(0) = T_1(0)\otimes T_2(0) = 0$

Again, (2.2) $\Rightarrow \|Tu\| \le c_1 c_2 \|u\|^2$.
So, $\|T^2 u\| = \|T(Tu)\| \le c_1 c_2 \|Tu\|^2 \le c_1 c_2 (c_1 c_2 \|u\|^2)^2 = kk^2\|u\|^4$, where $k = c_1 c_2$
Similarly, $\|T^3 u\| \le kk^2k^4\|u\|^8$, and $\|T^n u\| \le k^{2^n - 1}\|u\|^{2^n}$ ……… (2.3)
Now, $\|T^{n+1}u - T^n u\| = \|T(T^n u) - T(T^{n-1}u)\| \le k(\|T^n u\| + \|T^{n-1}u\|)^2$
$\to 0 \; as \; n \to \infty$ (using equation(2.3)), for $u \in B_{X\otimes_\gamma Y}$.
So, T is asymptotically regular at every point $u \in B_{X\otimes_\gamma Y}$.

Now similar to the part (ii) of the Lemma 2.1, $\{T^n u\}$ is a Cauchy sequence in $B_{X\otimes_\gamma Y}$ and so it converges to some $p \in B_{X\otimes_\gamma Y}$. Again, $\|p - T(p)\| = \|\lim_{n\to\infty} T^n u - T(\lim_{n\to\infty} T^n u)\| = \lim_{n\to\infty}\|T^n u - T^{n+1}u\| = 0$.Hence $p = T(p)$. Thus p is a fixed point of Tin $B_{X\otimes_\gamma Y}$.

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To show the uniqueness, let$q$ be the another fixed point of the operator Tin $B_{X\otimes_\gamma Y}$. Then $q = T(q)$. Now by part (ii) of Lemma 2.1, every sequence $\{T^n u\}$, where $u \in B_{X\otimes_\gamma Y}$ converges to the point$p\in B_{X\otimes_\gamma Y}$. So in particular, the sequence $\{T^n q\}$ is also converges to p. But $T^n q = T^{n-1}(Tq) = T^{n-1}q = \cdots = Tq = q$. Therefore, $\lim_{n\to\infty} T^n q = q = p$.

Thus, the fixed point p is unique and since $T(0) = 0$, so, obviously, 0 is this unique fixed point.□

Example 2.3 We have $l^1\otimes_\gamma X = l^1(X)$ (refer to [10]).Taking $X = \mathbb{K}$, we consider the mappings $T_1: l^1\otimes_\gamma \mathbb{K} \longrightarrow l^1$ defined by $T_1(\sum_i a_i\otimes x_i) = \frac{1}{2}\sum_i \{a_{i_n} x_i\}_n$, where $a_i = \{a_{i_n}\}_n$ and $T_2: l^1\otimes_\gamma \mathbb{K} \longrightarrow \mathbb{K}$ defined by $T_2(\sum_i a_i\otimes x_i) = \frac{1}{3}\sum_i \|a_i\||x_i|$. Clearly T_1 and T_2 are contraction mappings such that $T_1(0) = 0$ and $T_2(0) = 0$. So in $B_{l^1\otimes_\gamma \mathbb{K}}$, $T: l^1\otimes_\gamma \mathbb{K} \longrightarrow l^1\otimes_\gamma \mathbb{K}$ defined by $T(\sum_i a_i\otimes x_i) = \left(\frac{1}{2}\sum_i \{a_{i_n} x_i\}_n\right) \otimes \left(\frac{1}{3}\sum_i \|a_i\||x_i|\right)$ has a unique fixed point at 0.

Theorem 2.4 Let $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$ be two mappings satisfying

(i)$\|T_1 u - T_1 v\| \le k_1(\|F_1 u - T_1 u\| + \|F_1 v - T_1 v\|)$, $0 < k_1 < \frac{1}{2}$, for some $F_1: X\otimes_\gamma Y \longrightarrow X$ defined by $F_1(\sum_i x_i\otimes y_i) = \sum_i x_i\, g(y_i)$, $g \in Y^*$(the dual space of Y) with $\|g\| = 1$.
(ii)$\|T_2 u - T_2 v\| \le k_2(\|F_2 u - T_2 u\| + \|F_2 v - T_2 v\|)$, $0 < k_2 < \frac{1}{2}$, for some $F_2: X\otimes_\gamma Y \longrightarrow X$ defined by $F_2(\sum_i x_i\otimes y_i) = \sum_i f(x_i)\, y_i$, $f \in X^*$(the dual space of X) with $\|f\| = 1$,
where $u, v \in X\otimes_\gamma Y$.
(iii) $0\in$ Ker $T_1\cap$ KerT_2.

Then the operator $T: X\otimes_\gamma Y \longrightarrow X\otimes_\gamma Y$ defined by $Tu = T_1 u\otimes T_2 u$, $u \in X\otimes_\gamma Y$satisfies: $\|Tu - Tv\| \le c_1 c_2(\|u\| + \|v\|)^2$, where $c_1 c_2 < 1$, and has a unique fixed point at 0 in $B_{X\otimes_\gamma Y}$.

Proof.$\|Tu - Tv\| \le \|T_1 u - T_1 v\|\|T_2 v\| + \|T_1 u\|\|T_2 u - T_2 v\|$

$$\le k_1(\|F_1 u - T_1 u\| + \|F_1 v - T_1 v\|)k_2\|F_2 v - T_2 v\| + k_1\|F_1 u - T_1 u\|k_2(\|F_2 u - T_2 u\| + \|F_2 v - T_2 v\|)$$

$$\le k_1 k_2 \left[\begin{array}{r} (\|F_1 u\| + \|T_1 u\|)(\|F_2 u\| + \|T_2 u\|) + 2(\|F_1 u\| + \|T_1 u\|)(\|F_2 v\| + \|T_2 v\|) \\ +(\|F_1 v\| + \|T_1 v\|)(\|F_2 v\| + \|T_2 v\|) \end{array} \right] \ldots\ldots\ldots (2.4)$$

Now $\|T_1 u\| \le k_1\|F_1 u - T_1 u\| \le k_1(\|F_1 u\| + \|T_1 u\|) \le k_1(\|u\| + \|T_1 u\|)$, as $\|F_1\| \le 1$

$\Rightarrow \|T_1 u\| \le \frac{k_1}{1 - k_1}\|u\| = c_1\|u\|$, where $\frac{k_1}{1 - k_1} = c_1 < 1$ as $k_1 < \frac{1}{2}$

Similarly,$\|T_1 v\| \le c_1\|v\|$,$\|T_2 u\| \le c_2\|u\|$, where $c_2 < 1$and $\|T_2 v\| \le c_2\|v\|$.
Now, (2.4) $\Rightarrow$

$$\|Tu - Tv\| \le k_1 k_2(1 + c_1)(1 + c_2)[\|u\|^2 + 2\|u\|\|v\| + \|v\|^2] = k_1 k_2 \left(\frac{1}{1 - k_1}\right)\left(\frac{1}{1 - k_2}\right)(\|u\| + \|v\|)^2$$

$$= c_1 c_2(\|u\| + \|v\|)^2 \ldots\ldots\ldots (2.5)$$

Proceeding as in Theorem 2.2, we can show that T is asymptotically regular at every point in$B_{X\otimes_\gamma Y}$, and has a unique fixed point at 0in $B_{X\otimes_\gamma Y}$.□

Example 2.5 Let $T_1: l^1\otimes_\gamma \mathbb{R} \longrightarrow l^1$be $T_1(\sum_i a_i\otimes x_i) = \frac{1}{4}\sum_i \{a_{i_n} x_i\}$, where $a_i = \{a_{i_n}\}_n$ and $T_2: l^1\otimes_\gamma \mathbb{R} \longrightarrow \mathbb{R}$ be $T_2(\sum_i a_i\otimes x_i) = \frac{1}{4}\sum_i (sup_n a_{i_n})\, x_i$. Let $g \in \mathbb{R}^*$ be defined by
$g(x_i) = x_i$and$f \in (l^1)^*$be defined by $f(\{a_{i_n}\}) = sup_n a_{i_n}$.
For$s = \sum_i a_i\otimes x_i$, $t = \sum_i b_i\otimes y_i \in l^1\otimes_\gamma \mathbb{R}$, $\|T_1 s - T_1 t\| = \left\|\frac{1}{4}\sum_i \{a_{i_n} x_i\} - \frac{1}{4}\sum_i \{b_{i_n} y_i\}\right\|$

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$$\leq \frac{1}{3}\left[\left\|\left(\sum_i\{a_{i_n}\}x_i - \frac{1}{4}\sum_i\{a_{i_n}x_i\}\right)\right\| + \left\|\left(\sum_i\{b_{i_n}\}y_i - \frac{1}{4}\sum_i\{b_{i_n}y_i\}\right)\right\|\right]$$

$$\leq \frac{1}{3}\left[\left\|\left(\sum_i\{a_{i_n}\}g(x_i) - \frac{1}{4}\sum_i\{a_{i_n}x_i\}\right)\right\| + \left\|\left(\sum_i\{b_{i_n}\}g(y_i) - \frac{1}{4}\sum_i\{b_{i_n}y_i\}\right)\right\|\right]$$

$\leq \frac{1}{3}[\|F_1 s - T_1 s\| + \|F_1 t - T_1 t\|],\ k_1 = \frac{1}{3} < \frac{1}{2}$

Again, $\|T_2 s - T_2 t\| = \left\|\frac{1}{4}\sum_i(sup_n a_{i_n})\,x_i - \frac{1}{4}\sum_i(sup_n b_{i_n})\,y_i\right\|$

$$= \frac{1}{3}\left[\left\|\left(\sum_i(sup_n a_{i_n})\,x_i - \frac{1}{4}\sum_i(sup_n a_{i_n})\,x_i\right) - \left(\sum_i(sup_n b_{i_n})\,y_i - \frac{1}{4}\sum_i(sup_n b_{i_n})\,y_i\right)\right\|\right]$$

$$\leq \frac{1}{3}\left[\left\|\sum_i f(a_{i_n})\,x_i - \frac{1}{4}\sum_i(sup_n a_{i_n})\,x_i\right\| + \left\|\sum_i f(b_{i_n})\,y_i - \frac{1}{4}\sum_i(sup_n b_{i_n})\,y_i\right\|\right]$$

$= \frac{1}{3}[\|F_2 s - T_2 s\| + \|F_2 t - T_2 t\|]\ ,\ k_2 = \frac{1}{3} < \frac{1}{2}$

Also $T_1(0) = 0$and $T_2(0) = 0$. Therefore $T: l^1\otimes_\gamma \mathbb{R} \longrightarrow l^1\otimes_\gamma \mathbb{R}$ defined by $T(\sum_i a_i\otimes x_i) = \frac{1}{16}\sum_i\{Ma_{i_n}x_i\}_n$, where $M = \sum_i(sup_n a_{i_n})\,x_i$ has a unique fixed point at 0in $B_{l^1\otimes_\gamma\mathbb{R}}$.□

Modifying the conditions (i) and (ii) of Theorem 2.4, we get the following results.

Theorem 2.6 Let $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$ be two mappings satisfying

(i)$\|T_1 u - T_1 v\| \leq k_1(\|F_1 u - T_1 v\| + \|F_1 v - T_1 u\|),\ 0 < k_1 < \frac{1}{2}$
(ii)$\|T_2 u - T_2 v\| \leq k_2(\|F_2 u - T_2 v\| + \|F_2 v - T_2 u\|),\ 0 < k_2 < \frac{1}{2}$
(iii) $0\in$ Ker $T_1\cap$ KerT_2.

Then the operator T as defined earliersatisfies $\|Tu - Tv\| \leq c_1c_2(\|u\| + \|v\|)^2$, where $c_1c_2 < 1$, andhas a unique fixed point at 0 in $B_{X\otimes_\gamma Y}$.

Proof.$\|Tu - Tv\| \leq k_1(\|F_1 u - T_1 v\| + \|F_1 v - T_1 u\|)\frac{k_2}{1-k_2}\|v\| + \frac{k_1}{1-k_1}\|u\|k_2(\|F_2 u - T_2 v\| + \|F_2 v - T_2 u\|)$

$$\leq \frac{k_1k_2}{1-k_2}\left(\|F_1\|\|u\| + \frac{k_1}{1-k_1}\|v\| + \|v\| + \frac{k_1}{1-k_1}\|u\|\right)\|v\|$$
$$+ \frac{k_1k_2}{1-k_1}\left(\|F_2\|\|u\| + \frac{k_2}{1-k_2}\|v\| + \|v\| + \frac{k_2}{1-k_2}\|u\|\right)\|u\|$$
$$\leq \left(\frac{k_1}{1-k_1}\right)\left(\frac{k_2}{1-k_2}\right)(\|u\| + \|v\|)^2 = c_1c_2(\|u\| + \|v\|)^2$$
$$\text{where } c_1 = \frac{k_1}{1-k_1} \text{and } c_2 = \frac{k_2}{1-k_2}$$

So, as in Theorem 2.2, we can show that T has a unique fixed point at 0in $B_{X\otimes_\gamma Y}$.□

Theorem 2.7 Let $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$ be two mappings satisfying

(i)$\|T_1 u - T_1 v\| \leq a_1\|F_1 u - T_1 u\| + b_1\|F_1 v - T_1 v\| + c_1\|u - v\|, 2a_1 + 2b_1 + c_1 < 1$
(ii)$\|T_2 u - T_2 v\| \leq a_2\|F_2 u - T_2 u\| + b_2\|F_2 v - T_2 v\| + c_2\|u - v\|, 2a_2 + 2b_2 + c_2 < 1$
(iii) $0\in$ Ker $T_1\cap$ KerT_2.

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Then the operator T satisfies $\|Tu - Tv\| \le \alpha_1\alpha_2\|u\|^2 + 2\alpha_1\beta_2\|u\|\|v\| + \beta_1\beta_2\|v\|^2$, (where $\alpha_1, \alpha_2, \beta_1, \beta_2$ depend on a_1, b_1, c_1 and a_2, b_2, c_2),and has a unique fixed point at 0 in $B_{X\otimes_\gamma Y}$.

Proof. $\|Tu - Tv\| \le [a_1\|F_1u - T_1u\| + b_1\|F_1v - T_1v\| + c_1\|u - v\|]\frac{b_2+c_2}{1-b_2}\|v\| + \frac{a_1+c_1}{1-a_1}\|u\|[a_2\|F_2u - T_2u\| + b_2\|F_2v - T_2v\| + c_2\|u - v\|]$

$$\le \frac{b_2+c_2}{1-b_2}\|v\|[a_1\|F_1u\| + a_1\|T_1u\| + b_1\|F_1v\| + b_1\|T_1v\| + c_1\|u\| + c_1\|v\|]$$
$$+ \frac{a_1+c_1}{1-a_1}\|u\|[a_2\|F_2u\| + a_2\|T_2u\| + b_2\|F_2v\| + b_2\|T_2v\| + c_2\|u\| + c_2\|v\|]$$
$$\le \left(\frac{a_1+c_1}{1-a_1}\right)\left(\frac{a_2+c_2}{1-a_2}\right)\|u\|^2 + 2\left(\frac{a_1+c_1}{1-a_1}\right)\left(\frac{b_2+c_2}{1-b_2}\right)\|u\|\|v\| + \left(\frac{b_1+c_1}{1-b_1}\right)\left(\frac{b_2+c_2}{1-b_2}\right)\|v\|^2$$
$$= \alpha_1\alpha_2\|u\|^2 + 2\alpha_1\beta_2\|u\|\|v\| + \beta_1\beta_2\|v\|^2$$

where

$$\alpha_1 = \frac{a_1+c_1}{1-a_1}, \alpha_2 = \frac{a_2+c_2}{1-a_2}, \beta_1 = \frac{b_1+c_1}{1-b_1} \text{ and } \beta_2 = \frac{b_2+c_2}{1-b_2}$$

As in Theorem 2.2, $\|T^n u\| \le k^{2^n-1}\|u\|^{2^n}$, where $k = \alpha_1\alpha_2$.
$\|T^{n+1}u - T^n u\| \le \alpha_1\alpha_2\|T^n u\|^2 + 2\alpha_1\beta_2\|T^n u\|\|T^{n-1}u\| + \beta_1\beta_2\|T^{n-1}u\|^2$
$\to 0$ as $n \to \infty$, for $u \in B_{X\otimes_\gamma Y}$.

Thus T has a unique fixed point at 0in $B_{X\otimes_\gamma Y}$. □

Theorem 2.8 Let $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$ be two mappings satisfying

(i) $\|T_1u - T_1v\| \le a_1(\|F_1u - T_1u\| + \|F_1v - T_1v\|) + b_1(\|F_1u - T_1v\| + \|F_1v - T_1u\|) + c_1\|u - v\|$,
where $2a_1 + 2b_1 + c_1 < 1$
(ii) $\|T_2u - T_2v\| \le a_2(\|F_2u - T_2u\| + \|F_2v - T_2v\|) + b_2(\|F_2u - T_2v\| + \|F_2v - T_2u\|) + c_2\|u - v\|$,
where $2a_2 + 2b_2 + c_2 < 1$
(iii) $0 \in \text{Ker } T_1 \cap \text{Ker} T_2$.

Then the operator T satisfies $\|Tu - Tv\| \le \alpha_1\alpha_2(\|u\| + \|v\|)^2$, where α_1, α_2 depend on a_1, b_1, c_1 and a_2, b_2, c_2, and has a unique fixed point at 0 in $B_{X\otimes_\gamma Y}$.

Proof.

$$\|Tu - Tv\| \le [a_1(\|F_1u - T_1u\| + \|F_1v - T_1v\|) + b_1(\|F_1u - T_1v\| + \|F_1v - T_1u\|) + c_1\|u - v\|]\frac{a_2+b_2+c_2}{1-a_2-b_2}\|v\|$$
$$+ \frac{a_1+b_1+c_1}{1-a_1-b_1}\|u\|[a_2(\|F_2u - T_2u\| + \|F_2v - T_2v\|) + b_2(\|F_2u - T_2v\| + \|F_2v - T_2u\|)$$
$$+ c_2\|u - v\|]$$
$$\le \frac{a_2+b_2+c_2}{1-a_2-b_2}\|v\|[a_1(\|F_1\|\|u\| + \|T_1u\|) + a_1(\|F_1\|\|v\| + \|T_1v\|) + b_1(\|F_1\|\|v\| + \|T_1u\|)$$
$$+ b_1(\|F_1\|\|u\| + \|T_1v\|) + c_1\|u\| + c_1\|v\|]$$
$$+ \frac{a_1+b_1+c_1}{1-a_1-b_1}\|u\|[a_2(\|F_2\|\|u\| + \|T_2u\|) + a_2(\|F_2\|\|v\| + \|T_2v\|) + b_2(\|F_2\|\|v\| + \|T_2u\|)$$
$$+ b_2(\|F_2\|\|u\| + \|T_2v\|) + c_2\|u\| + c_2\|v\|]$$

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$$\leq \frac{a_2+b_2+c_2}{1-a_2-b_2}\|v\|\left[(a_1+b_1+c_1)(\|u\|+\|v\|)+(a_1+b_1)\left(\frac{a_1+b_1+c_1}{1-a_1-b_1}\right)(\|u\|+\|v\|)\right]$$
$$+\frac{a_1+b_1+c_1}{1-a_1-b_1}\|u\|\left[(a_2+b_2+c_2)(\|u\|+\|v\|)+(a_2+b_2)\left(\frac{a_2+b_2+c_2}{1-a_2-b_2}\right)(\|u\|+\|v\|)\right]$$
$$=\left(\frac{a_1+b_1+c_1}{1-a_1-b_1}\right)\left(\frac{a_2+b_2+c_2}{1-a_2-b_2}\right)\big((\|u\|+\|v\|)\|v\|+(\|u\|+\|v\|)\|u\|\big)=\alpha_1\alpha_2(\|u\|+\|v\|)^2$$

where, $\alpha_1=\frac{a_1+b_1+c_1}{1-a_1-b_1}<1$, $\alpha_2=\frac{a_2+b_2+c_2}{1-a_2-b_2}<1$

So, proceeding as in Theorem 2.2, we can show that T has a unique fixed point at 0 in $B_{X\otimes_\gamma Y}$. □

Theorem 2.9 Let $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$ be two mappings satisfying

(i) $\|T_1u-T_1v\| \leq h_1 \max\{\|F_1u-T_1u\|, \|F_1v-T_1v\|\}$, $0<h_1<\frac{1}{2}$
(ii) $\|T_2u-T_2v\| \leq h_2 \max\{\|F_2u-T_2u\|, \|F_2v-T_2v\|\}$, $0<h_2<\frac{1}{2}$
(iii) $0\in$ Ker $T_1\cap$ KerT_2.

Then the operator T satisfies $\|Tu-Tv\| \leq \alpha_1\alpha_2 \max\{\|u\|,\|v\|\}(\|u\|+\|v\|)$, (where α_1,α_2 depend on h_1h_2), and has a unique fixed point at 0 in $B_{X\otimes_\gamma Y}$.

Proof. $\|Tu-Tv\| \leq h_1 \max\{\|F_1u-T_1u\|, \|F_1v-T_1v\|\} h_2 \|F_2v-T_2v\|$
$$+h_1\|F_1u-T_1u\| h_2 \max\{\|F_2u-T_2u\|,\|F_2v-T_2v\|\}$$
$$\leq h_1h_2(\|F_2v\|+\|T_2v\|)\max\{\|F_1u-T_1u\|,\|F_1v-T_1v\|\}$$
$$+\ h_1h_2(\|F_1u\|+\ \|T_1u\|)\max\{\|F_2u-\ T_2u\|,\|F_2v-T_2v\|\}$$
$$\leq h_1h_2\left(\|F_2\|\|v\|+\frac{h_2}{1-h_2}\|v\|\right)\max\left\{\left(\|F_1\|\|u\|+\frac{h_1}{1-h_1}\|u\|\right),\left(\|F_1\|\|v\|+\frac{h_1}{1-h_1}\|v\|\right)\right\}$$
$$+h_1h_2\Big(\|F_1\|\|u\|$$
$$+\frac{h_1}{1-h_1}\|u\|\Big)\max\left\{\left(\|F_2\|\|u\|+\frac{h_2}{1-h_2}\|u\|\right),\left(\|F_2\|\|v\|+\frac{h_2}{1-h_2}\|v\|\right)\right\}$$
$$=\left(\frac{h_1}{1-h_1}\right)\left(\frac{h_2}{1-h_2}\right)\max\{\|u\|,\|v\|\}(\|u\|+\|v\|)=\alpha_1\alpha_2\max\{\|u\|,\|v\|\}(\|u\|+\|v\|)$$
where $\alpha_1=\frac{h_1}{1-h_1}<1, \alpha_2=\frac{h_1}{1-h_1}<1.$

For $v=0$, $\|T(u)\|\leq\alpha_1\alpha_2\|u\|^2$. So, proceeding as in Theorem 2.2, we can show that T has a unique fixed point at 0 in $B_{X\otimes_\gamma Y}$. □

Remark 2.10 If T_1 and T_2 are such that $T_1(0)=\lambda_1\neq 0\in X$ and $T_2(0)=\lambda_2\neq 0\in Y$, then we can take $\widetilde{T_1}: X\otimes_\gamma Y \longrightarrow X$ such that $\widetilde{T_1}(u)=T_1(u)-\lambda_1$ and $\widetilde{T_2}: X\otimes_\gamma Y \longrightarrow Y$ such that $\widetilde{T_2}(u)=T_2(u)-\lambda_2$. Therefore $\widetilde{T_1}(0)=0$ and $\widetilde{T_2}(0)=0$. So $T: X\otimes_\gamma Y \longrightarrow X\otimes_\gamma Y$ defined by $(u)=\widetilde{T_1}(u)\otimes\widetilde{T_2}(u)$, and all the above Theorems are valid for T.

Deduction 2.11 In Theorem 2.2, if we take T_2 to be non-expansive, then also we get analogous result.

Proof. For $u,v\in X\otimes_\gamma Y$,
$\|T_1u-T_1v\|\leq k_1\|u-v\|$, where $0\leq k_1<1$, and $\|T_2u-T_2v\|\leq\|u-v\|$
$$\|Tu\|=\|T_1u\otimes T_2u\|=\|T_1u\|\|T_2u\|\leq k_1\|u\|\|u\|=k_1\|u\|^2$$
Therefore, $\|T^nu\|\leq k_1\|T^{n-1}u\|^2\leq\cdots\leq {k_1}^{2^n-1}\|u\|^{2^n}\rightarrow 0$ as $n\rightarrow\infty$ if $u\in B_{X\otimes_\gamma Y}$

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Now, $\|T^{n+1}u - T^n u\| \leq \|T^{n+1}u\| + \|T^n u\| \to 0$ as $n \to \infty$ for $u \in B_{X\otimes_\gamma Y}$.
So, T is asymptotically regular at every point $u \in B_{X\otimes_\gamma Y}$ and thus T has a unique fixed point at 0 in $B_{X\otimes_\gamma Y}$. □

All the Theorems 2.4-2.9 are true taking T_2 as a non-expansive mapping.

Now, we study the converse problem of Theorem 2.2, i.e., given a contraction mapping with some fixed point on the space $X\otimes_\gamma Y$, can we construct some contraction mappings for each of the spaces X and Y with some fixed points? Here, we give an affirmative answer to this problem by the following result.

Theorem 2.12 Let $T: X\otimes_\gamma Y \longrightarrow X\otimes_\gamma Y$ be a contraction mapping having the unique fixed point $\alpha\otimes\beta$, where α lies on the unit sphere $S_X(i.e. \|\alpha\| = 1)$ and β lies on the unit sphere $S_Y(i.e. \|\beta\| = 1)$. Then corresponding to T, there exist contraction mappings S_1 on X and S_2 on Y such that α and β are the fixed points of S_1 and S_2 respectively.

Proof. Since $\|\alpha\| = 1$, so $\alpha \neq 0$. Hence there exists some $f \in X^*$ such that $f(\alpha) = \|\alpha\|$and $\|f\| = 1$. Similarly, since $\|\beta\| = 1$, there exists some $g \in Y^*$ such that $g(\beta) = \|\beta\|$and $\|g\| = 1$.
Now we define two linear maps $F_1: X\otimes_\gamma Y \longrightarrow X$by $F_1(\sum_i x_i\otimes y_i) = \sum_i x_i\, g(y_i)$, and $F_2: X\otimes_\gamma Y \longrightarrow Y$ by $F_2(\sum_i x_i\otimes y_i) = \sum_i f(x_i)\, y_i$. Then $\|F_1\| \leq \|g\|$, $\|F_2\| \leq \|f\|$.
Let $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$ be defined by $T_1(\sum_i x_i\otimes y_i) = F_1\big(T(\sum_i x_i\otimes y_i)\big)$ and $T_2(\sum_i x_i\otimes y_i) = F_2\big(T(\sum_i x_i\otimes y_i)\big)$.
For$u, v \in X\otimes_\gamma Y$,
$\|T_1 u - T_1 v\| = \|F_1(Tu) - F_1(Tv)\| \leq \|F_1\|\|Tu - Tv\| \leq \|g\|k\|u - v\| = k\|u - v\|$, where $0 \leq k < 1$ (as T is a contraction).

Therefore T_1 is a contraction. Similarly T_2 is also a contraction.

Now, we define $S_1: X \longrightarrow X$ be such that $S_1(x) = T_1(x\otimes\beta)$, $x \in X$ and
$S_2: Y \longrightarrow Y$be such that $S_2(y) = T_2(\alpha\otimes y)$, $y \in Y$. Then
$\|S_1(x) - S_1(x')\| \leq \|T_1(x\otimes\beta) - T_1(x'\otimes\beta)\| \leq k\|x\otimes\beta - x'\otimes\beta\| = k\|x - x'\|\|\beta\| = k\|x - x'\|$, $x, x' \in X$
and $\|S_2(y) - S_2(y')\| \leq k\|\alpha\otimes y - \alpha\otimes y'\| = k\|\alpha\|\|y - y'\| = k\|y - y'\|$, $y, y' \in Y$

Thus S_1 and S_2 are contraction mappings and so, have the unique fixed points in X and Y respectively. Now,$S_1(\alpha) = T_1(\alpha\otimes\beta) = F_1\big(T(\alpha\otimes\beta)\big) = F_1(\alpha\otimes\beta) = \alpha g(\beta) = \alpha\|\beta\| = \alpha$and
$S_2(\beta) = T_2(\alpha\otimes\beta) = F_2\big(T(\alpha\otimes\beta)\big) = F_2(\alpha\otimes\beta) = f(\alpha)\beta = \|\alpha\|\beta = \beta$

Therefore α and β are the unique fixed points of S_1 and S_2 respectively. □

From two contraction mappings S_1 and S_2 (with fixed points) on the Banach spaces X and Y respectively, the mapping T on $X\otimes_\gamma Y$ can be constructed easily with a fixed point.

Theorem 2.13 Let $S_1: X \longrightarrow X$ and $S_2: Y \longrightarrow Y$ be two contraction mappings with the fixed points α and β lying onS_X and S_Y respectively. Then using S_1 and S_2 we can construct themapping$T$ on $X\otimes_\gamma Y$ with the unique fixed point $\alpha\otimes\beta$in $B_{X\otimes_\gamma Y}$.

Proof. Let F_1 and F_2be two linear maps as defined in the previous theorem. We define $T_1: X\otimes_\gamma Y \longrightarrow X$ by $T_1(u) = S_1\big(F_1(u)\big)$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$be defined $T_2(u) = S_2\big(F_2(u)\big)$ where $u \in X\otimes_\gamma Y$. Now

$$\|T_1 u - T_1 v\| = \big\|S_1\big(F_1(u)\big) - S_1\big(F_1(v)\big)\big\| \leq k_1\|F_1 u - F_1 v\| \leq k_1\|F_1\|\|u - v\| \leq k_1\|g\|\|u - v\|$$
$$= k_1\|u - v\|, \text{ as } \|g\| = 1$$

where$0 \leq k_1 < 1$, as S_1 is contraction.

Therefore T_1 is contraction. Similarly T_2 is also a contraction. Now

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$T: X\otimes_\gamma Y \longrightarrow X\otimes_\gamma Y$ is such that $Tu = T_1u\otimes T_2u$.

$$T(\alpha\otimes\beta) = T_1(\alpha\otimes\beta)\otimes T_2(\alpha\otimes\beta) = S_1\big(F_1(\alpha\otimes\beta)\big)\otimes S_2\big(F_2(\alpha\otimes\beta)\big)$$
$$= S_1\big(\alpha g(\beta)\big)\otimes S_2\big(f(\alpha)\beta\big) \text{(by definition of } f \text{ and } g \text{ given in theorem 2.12)}$$
$$= S_1(\alpha\|\beta\|)\otimes S_2(\|\alpha\|\beta) = S_1(\alpha)\otimes S_2(\beta) = \alpha\otimes\beta$$

Clearly, $\alpha\otimes\beta \in B_{X\otimes_\gamma Y}$ is the unique fixed point of T in $B_{X\otimes_\gamma Y}$.□

III. CONCLUSION

We have discussed different fixed point theorems with different contractive type mappings on tensor product spaces. Moreover, using a given contraction mapping (with fixed point) on the tensor product space $X\otimes_\gamma Y$, we have constructed some contraction mappings with fixed points for the individual spaces X and Y. However, many other open problems can be raised regarding different types of contractive mappings on tensor product spaces. In [1], Alber and Guerre-Delabriere defined weakly contractive maps. In [12], Rhoades extended some results on weakly contractive maps to arbitrary Banach spaces.For a Banachspace X, and a closed convex subset K of X, a self-map T of K is called weakly contractive if for each $x, y \in K$,

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|)$$

where $\psi: [0,\infty) \longrightarrow [0,\infty)$ is continuous and non-decreasing such that ψ is positive on $(0,\infty)$, $\psi(0) = 0$, and $\lim_{t\to\infty} \psi(t) = \infty$. If K is bounded, then the infinity condition can be omitted.

Now we can raise the following problem:
Given two weakly contractive maps $T_1: X\otimes_\gamma Y \longrightarrow X$ and $T_2: X\otimes_\gamma Y \longrightarrow Y$, can we obtain some fixed point theorems for the mapping T on $X\otimes_\gamma Y$?

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