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Efficient Domination number and Chromatic number of a Fuzzy Graph

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Abstract: A subset S of V is called a domination set in G if every vertex in V-S is adjacent to at least one vertex in S. A dominating set is said to be Fuzzy Total Dominating set if every vertex in V is adjacent to at least one vertex in S. Minimum cardinality taken over all total dominating set is called as fuzzy total domination number and is denoted by γ_{ft}(G). The minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour is the chromatic number χ(G). For any graph G a complete sub graph of G is called a clique of G. In this paper we find an upper bound for the sum of the fuzzy total domination and chromatic number in fuzzy graphs and characterize the corresponding extremal fuzzy graphs.

Keyword: Fuzzy Efficient Domination Number, Chromatic Number, Clique, Fuzzy Graphs

I.INTRODUCTION

Let G(μ,σ) be simple undirected fuzzy graph. The degree of any vertex u in G is the number of edges incident with u and is denoted by d(u). The minimum and maximum degree of a vertex is denoted by δ(G) and Δ(G) respectively, P_n denotes the path on n vertices. The vertex connectivity κ(G) of a graph G is the minimum number of vertices whose removal results in a disconnected graph. The chromatic number χ is defined to be the minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour. For any graph G a complete sub graph of G is called a clique of G. The number of vertices in a largest clique of G is called the clique number of G.

A subset S of V is called a dominating set in G, if every vertex in V-S is adjacent to at least one vertex is S. The minimum cardinality taken over all minimal dominating sets in G is called the domination number of G and is denoted by γ. A dominating set S is said to be fuzzy total dominating set if every vertex in V is adjacent to at least one vertex in S. Minimum cardinality taken over all total dominating set is called as fuzzy total domination number and is denoted by γ_{ft}(G).

If X is collection of objects denoted generically by x, then a Fuzzy set $\tilde{A}$ is X is a set of ordered pairs: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))/x \in X\}$, $\mu_{\tilde{A}}(x)$ is called the membership function of x in $\tilde{A}$ that maps X to the membership space M (when M contains only the two points 0 and 1). Let E be the (crisp) set of nodes. A fuzzy graph is then defined by, $\tilde{G}(x_i,x_j) = \{(x_i,x_j), \mu_{\tilde{G}}(x_i,x_j)/(x_i,x_j) \in E \times E\}$. $\tilde{H}(x_i,x_j)$ is a Fuzzy Sub graph of $\tilde{G}(x_i,x_j)$ if $\mu_{\tilde{H}}(x_i,x_j) \le \mu_{\tilde{G}}(x_i,x_j)\ \forall (x_i,x_j) \in E \times E$, $\tilde{H}(x_i,x_j)$ is a spanning fuzzy sub graph of $\tilde{G}(x_i,x_j)$ if the node set of $\tilde{H}(x_i,x_j)$ and $\tilde{G}(x_i,x_j)$ are equal, that is if they differ only in their arc weights.

The fuzzy definition of fuzzy graphs was proposed by Kaufmann [4], from the fuzzy relations introduced by Zadeh [9]. Although Rosenfeld [5] introduced another elaborated definition, including fuzzy vertex and fuzzy edges. Several fuzzy analogs of graph theoretic concepts such as paths, cycles connectedness etc. The concept of domination in fuzzy graphs was investigated by A.Somasundram, S.Somasundram [6]. A. Somasundram presented the concepts of independent domination, total domination, connected domination and domination in cartesian product and composition of fuzzy graphs([7][8]).

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [10], Paulraj Joseph J and Arumugam S proved that γ+k≤p. In[9], Paulraj Joseph J and Arumugam S proved that γ_c (G)+ $\chi \leq$ p+1. They also characterized the class of graphs for which the upper bound is attained. They also proved similar results for γ and γ_t. In[14], Mahadevan G introduced the concept the complementary perfect domination number γ_{cp} and proved that γ_{cp}(G)+ $\chi \leq$ 2n-2, and characterized the corresponding extermal graphs. In[15], S.Vimala and J.S.Sathya proved that γ_t (G)+χ(G)=2n-5. They also characterised the class of graphs for which the upper bound is attained. In this paper we obtain sharp upper bound for the sum of the fuzzy Efficient domination number and chromatic number and characterize the corresponding extremal fuzzy graphs. We use the following previous results.

Theorem 1.1 [1]: For any connected graph G, γ_e (G) $\leq n$

Theorem 1.2 [2]: For any connected graph G, χ(G)≤ Δ(G)+1.

II. MAIN RESULTS

Theorem 2.1: For any connected fuzzy graph G, γ_e(G)+ χ(G)≤2n and the equality holds if and only if G≅K_1

Proof: γ_e(G)+χ(G) ≤ n+Δ+1=n+(n-1)+1≤2n. If γ_e (G)+χ(G)=2n the only possible case is γ_e(G)=n and χ(G)=n, Since χ(G)=n, G=K_n , But for K_n, γ_e (G)=1, so that G≅K_1. Converse is obvious.

Theorem 2.2: For any connected fuzzy graph G, γ_e (G)+χ(G)=2n-1 and the equality holds if and only if G≅K_2

Proof: Assume that γ_e (G)+χ(G)=2n-1. This is possible only if γ_e (G)=n and χ(G)=n-1 (or) γ_e (G)=n-1 and χ(G)=n.

Case (i) Let γ_e (G)=n and χ(G)=n-1.

Since χ(G)= n-1, G contains a clique K on n-1 vertices. Let x be a vertex of G-K_{n-1}. Since G is connected the vertex x is adjacent to one vertex u_i for some i in K_{n-1} {u_i} is γ_e – set, so that γ_e(G)=1, we have n=1. Then $\chi = 0$, which is a contradiction. Hence no fuzzy graph exists.

Case (ii) Let γ_e (G)=n-1 and χ(G)=n

Since χ(G)=n, G=K_n , But for K_n , γ_e(G)=1, so that n=2, χ =2 Hence G≅K_2. Converse is obvious.

Theorem 2.3: For any connected fuzzy graph G, γ_e (G)+χ(G)=2n-2 and the equality holds if and only if G≅K_3

Proof: Assume that γ_e (G)+χ(G)=2n-2. This is possible only if γ_e (G)=n and χ(G)=n-2 (or) γ_e (G)=n-1 and χ(G)=n-1 (or) γ_e (G)=n-2 and χ(G)=n.

Case (i) Let γ_e (G)=n and χ(G)=n-2.

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Since χ (G)= n-2, G contains a clique K on n-2 vertices. Let S={x,y}∈V-S. Then $< S >= K_2$ or $\overline{K_2}$

Subcase (a) Let $< S >= K_2$ Since G is connected, x is adjacent to some u_i of K_{n-2}. Then $\{x,u_j\}$ is γ_e – set for some $i \neq j$, so that γ_e (G)=2 and hence n=2. But χ(G)=n-2=0. Which is a contradiction. Hence no fuzzy graph exists.

Subcase (b) Let $< S >= \overline{K_2}$ Since G is connected, x is adjacent to some u_i of K_{n-2}. Then y is adjacent to the same u_i of $K_{n-2.}$ Then $\{u_j,x,y\}$ γ_e - set, so that γ_e(G)=1 and hence n=1. But χ(G)=n-2=negative value. Which is a contradiction. Hence no fuzzy graph exists, or y is adjacent to u_j of K_{n-2} for i≠j. In this case $\{u_i,y\}$ γ_e - set, so that γ_e (G)=2 and hence n=2. But χ(G)=0. Which is a contradiction. Hence no fuzzy graph exists.

Case (ii) Let γ_e (G)=n-1 and χ(G)=n-1.

Since χ (G)= n-1, G contains a clique K on n-1 vertices. Let x be a vertex of G-K_{n-1}. Since G is connected, x is adjacent to one vertex u_i for some i in K_{n-1}, so that $\{u_i\}$is γ_e -set of G γ_e (G) = 1, we have n=2. Then $\chi = 1$, which is a contradiction. Hence no fuzzy graph exists.

Case (iii) Let γ_e (G)=n-2 and χ(G)=n

Since χ(G)=n, G=K_n , But for K_n , γ_e (G) = 1, so that n=3, χ = 3 Hence G≅K_3. Converse is

obvious.

Theorem 2.4: For any connected fuzzy graph G, γ_e (G)+ χ(G)=2n-3 and the equality holds if and only if G≅P_3,K_4

Proof: Assume that γ_e (G)+ χ(G)=2n-3. This is possible only if γ_e (G)=n and χ(G)=n-3 (or) γ_e (G)=n-1 and χ(G)=n-2 (or) γ_e (G)=n-2 and χ(G)=n-1(or) γ_e (G)=n-3 and χ(G)=n.

Case (i) Let γ_e (G)=n and χ(G)=n-3.

Since χ (G)= n-3, G contains a clique K on n-3 vertices. Let S={x,y,z}∈V-S. Then $< S >= K_3$, $\overline{K_3}$, K_2∪K_1,P_3

Subcase (i) Let $< S >= K_3$. Since G is connected, x is adjacent to some u_i of K_{n-3}. Then $\{y,u_j\}$ is γ_e - set, so that γ_e (G)=2 and hence n=2. But χ(G)=n-3=negative value. Which is a contradiction. Hence no fuzzy graph exists.

Subcase (ii) Let $< S >= \overline{K_3}$ Since G is connected, one of the vertices of K_{n-3} say u_i is adjacent to all the vertices of S or two vertices of S or one vertex of S. If u_i for some i is adjacent to all the vertices of S, then $\{u_i\}$ in K_{n-3} is a γ_e -set of G, so that γ_e (G)=1 and hence n=1. But χ(G)=n-3=negative value. Which is a contradiction. Hence no fuzzy graph exists. Since G is connected u_i for some i is adjacent to two vertices of S say x and y and z is adjacent to u_j for i≠j in K_{n-3}, then $\{u_i,z\}$ in K_{n-3} is γ_e -set of G, so that γ_e (G)=2 and hence n=2. But χ(G)=n-3=negative value. Which is a contradiction. Hence no fuzzy graph exists. If u_i for some i is adjacent to x and u_j is adjacent to y and u_k is adjacent to z, then $\{u_i,y,z\}$ for i≠j≠k in K_{n-3} is a γ_e-set of G. so that γ_e (G)=3 and hence n=3. But χ(G)=n-3=0. Which is a contradiction. Hence no fuzzy graph exists.

Subcase (iii) Let $< S >= P_3 = \{x,y,z\}$. Since G is connected, x(or equivalently z) is adjacent to u_i for some i in K_{n-3}. Then $\{z,u_i\}$ is a γ_e -set of G. so that γ_e (G)=2 and hence n=2. But χ(G)=n-3 is the negative value. Which is a contradiction. Hence no fuzzy graph exists. If u_i is adjacent to y then $\{u_j,y\}$ is a γ_e -set of G. so that γ_e (G)=2 and hence n=2. But χ(G)=n-3=negative value. Which is a contradiction. Hence no fuzzy graph exists.

Subcase (iv) Let $< S >= K_2 \cup K_1$ Let xy be the edge and z be the isolated vertex of $K_2 \cup K_1$ Since G is connected, there exists a u_i in K_{n-3} is adjacent to x and z. Then $\{u_i,y\}$ is γ_e -set of G, so that γ_e (G)=2 and hence

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n=1. But χ(G)=n-3=negative value. Which is a contradiction. Hence no fuzzy graph exists. If z is adjacent to u_j for some i≠j then {y,u_j} for i≠j is γ_e -set of G, so that γ_e (G)=2 and hence n=2. But χ(G)=n-3=negative value. Which is a contradiction. Hence no fuzzy graph exists.

Case (ii) Let γ_e (G)=n-1 and χ(G)=n-2.

Since χ(G)=n-2, G contains a clique K on n-2 vertices. Let S={x,y}∈V-S. Then $< S >= K_2$ or $\overline{K_2}$

Subcase (a) Let $< S >= K_2$ Since G is connected, x(or equivalently y) is adjacent to some u_i of K_{n-2}. Then {y,u_j} for some i≠j is γ_e - set, so that γ_e(G)=2 and hence n=3. But χ(G)=n-2=1 for which G is totally disconnected, which is a contradiction. Hence no fuzzy graph exists.

Subcase (b) Let $< S >= \overline{K_2}$ Since G is connected, x is adjacent to some u_i of K_{n-2}. Then y is adjacent to the same u_i of $K_{n-2.}$ Then {u_i} is γ_e - set, so that γ_e (G)=1 and hence n=2. But χ(G)=n-2=0. Which is a contradiction. Hence no fuzzy graph exists. Otherwise x is adjacent to u_i of K_{n-2} for some i and y is adjacent to u_j of K_{n-2} for i≠j. In this {u_i,y} γ_t - set, so that γ_e (G)=2 and hence n=3. But χ(G)=1 for which G is totally disconnected. Which is a contradiction. In this case also no fuzzy graph exists.

Case (iii) Let γ_e (G)=n-2 and χ(G)=n-1.

Since χ(G)=n-1, G contains a clique K on n-1 vertices. Let x be a vertex of K_{n-1}. Since G is connected the vertex x is adjacent to one vertex u_i for some i in K_{n-1} so that {u_i} γ_e -set of G γ_e (G)=1, we have n=3 and $\chi = 2$ .Then K=K_2= uv. If x is adjacent to u_i, then G ≅ P_3.

Case (iv) Let γ_e (G)=n-3 and χ(G)=n

Since χ(G)=n, G=K_n, But for K_n, γ_e(G)=1, so that n=4, $\chi = 4$ Hence G≅K_4. Converse is obvious.

Theorem 2.5: For any connected fuzzy graph G, γ_e (G)+χ(G)=2n-4 and the equality holds if and only if G≅P_4,K_5 , K_4-e or the graph in figure 2.1

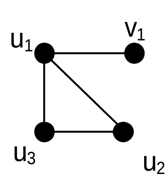

Figure 2.1

Proof: Assume that γ_e (G)+χ(G)=2n-4. This is possible only if γ_e (G)=n and χ(G)=n-4 (or) γ_e (G)=n-1 and χ(G)=n-3 (or) γ_e (G)=n-2 and χ(G)=n-2 (or) γ_e (G)=n-3 and χ(G)=n-1 (or) γ_e (G)=n-4 and χ(G)=n.

Case (i) Let γ_e (G)=n and χ(G)=n-4.

Since χ(G)=n-4, G contains a clique K on n-4 vertices. Let S = {v_1, v_2, v_3, v_4}. Then the induced subgraph <s> has the following possible cases K_4,$\overline{K}_4$,P_4,P_3UK_1,K_2UK_2,K_3UK_1,$K_{1,3}$

In all the above cases, it can be verified that no new fuzzy graphs exists.

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Case(ii) Let γ_e (G)=n-1 and χ(G)=n-3.

Since χ(G)=n-3, G contains a clique K on n-3 vertices. Let S={x,y,z}∈V-S. Then <S>=K_3 ,$\overline{K_3}$, $K_2 \cup K_1$,P_3

Subcase (i) Let $< S >= K_3$. Since G is connected, x is adjacent to some u_i of K_{n-3}. Then {x,u_j} for some i≠j is γ_e - set, so that γ_e(G)=2 and hence n=3. But χ(G)=n-3=0. Which is a contradiction. Hence no fuzzy graph exists.

Subcase (ii) Let $< S >= \overline{K_3}$ Since G is connected, one of the vertices of K_{n-3} say u_i is adjacent to all the vertices of S or two vertices of S or one vertex of S. If u_i for some i is adjacent to all the vertices of S, then {u_i} in K_{n-3} is γ_e -set of G. so that γ_e (G)=1 and hence n=2. But χ(G)=n-3=negative value. Which is a contradiction. Hence no fuzzy graph exists. If u_i for some i is adjacent to two vertices of S say x and y then G is connected, z is adjacent to u_j for i≠j in K_{n-3}, then {x,y,u_j} in K_{n-3} is γ_e -set of G, so that γ_e(G)=3 and hence n=4. But χ(G)=4-3=1 for which G is totally disconnected. Which is a contradiction. Hence no fuzzy graph exists. If u_i for some i is adjacent to x and u_j is adjacent to y and u_k is adjacent to z, then {x,z,u_j} for i≠j≠k in K_{n-3} is γ_e -set of G. so that γ_e (G)=3 and hence n=4. But χ(G)=1 for which G is totally disconnected. Which is a contradiction. Hence no fuzzy graph exists.

Subcase (iii) Let $< S >= P_3 = \{x, y, z\}$. Since G is connected, x(or equivalently z) is adjacent to u_i for some i in K_{n-3}. Then {z,u_i} is γ_e-set of G. so that γ_e (G)=2 and hence n=3. But χ(G)=n-3=1. Which is a contradiction. Hence no fuzzy graph exists. If u_i is adjacent to y then γ_e(G)=0 and hence n=1. But χ(G)=negative value. Which is a contradiction. Hence no fuzzy graph exists.

Subcase (iv) Let $< S >= K_2 \cup K_1$ Let xy be the edge and z be a isolated vertex of $K_2 \cup K_1$ Since G is connected, there exists a u_i in K_{n-3} is adjacent to x and z also adjacent to same u_i Then no γ_e - set exists. So that γ_e (G)=0 and χ(G)=n-3=negative value. Which is a contradiction. Hence no fuzzy graph exists. If z is adjacent to u_j for some i≠j then {y,u_j} for i≠j is a is a γ_e -set of G. so that γ_e (G)=2 and hence n=3. But χ(G)=n-3=0.Which is a contradiction. Hence no fuzzy graph exists.

Case (iii) Let γ_e (G)=n-2 and χ(G)=n-2.

Since χ (G)= n-2, G contains a clique K on n-2 vertices. Let S={x,y}∈V-S. Then $< S >= K_2$ or $\overline{K_2}$

Subcase (a) Let $< S >= K_2$. Since G is connected, x(or equivalently y) is adjacent to some u_i of K_{n-2}. Then {y,u_j} is γ_e - set, so that γ_e(G)=2 and hence n=4. But χ(G)=n-2=2. Then G≅P_4.

Subcase (b) Let $< S >= \overline{K_2}$, since G is connected, x is adjacent to some u_i of K_{n-2}. Then y is adjacent to the same u_i of K_{n-2}. Then {u_i} is γ_e - set, so that γ_e (G)=1 and hence n=3. But χ(G)=n-2=1.Which is a contradiction. Hence no fuzzy graph exists, or y is adjacent to u_j of K_{n-2} for i≠j. In this case no γ_e – set exists, so that γ_e (G)=0 and hence n=2. But χ(G)=0.Which is a contradiction. Hence no fuzzy graph exists.

Case (iv) Let γ_e (G)=n-3 and χ(G)=n-1.

Since χ (G)= n-1, G contains a clique K on n-1 vertices. Let x be a vertex of G-K_{n-1}. Since G is connected the vertex x is adjacent to one vertex u_i for some i in K_{n-1}, Then {u_i} is γ_e – set, so that γ_e(G) = 1, we have n=4 and $\chi = 3$. Then K=K_3 Let u_1,u_2,u_3 be the vertices of K_3. Then x must be adjacent to only one vertex of G-K_3. Without loss of generality let x be adjacent to u_1.If d(x)=1, then $G \cong G_1$.(in Fig 2.1). If d(x)=2 that is x is adjacent to u_2 or u_3 . Then γ_e(G) = 1, we have n=4 and $\chi = 3$. Then $G \cong K_4 - e$

Case (v) Let γ_e (G)=n-4 and χ(G)=n

Since χ(G)=n, G=K_n, But for K_n, γ_e(G)=1, so that n=5, $\chi = 5$. Hence G≅K_5. Converse is obvious.

III.CONCLUSION

In this paper, upper bound of the sum of Efficient domination and chromatic number is proved. In future this result can be extended to various domination parameters. The structure of the graphs had been given in this paper can be used in models and networks. The authors have obtained similar results with large cases of graphs for which γ_e(G)+χ (G)=2n-5, γ_e(G)+χ (G)=2n-6, γ_e(G)+χ (G)=2n-7

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