ISSN(Online): 2319-8753
ISSN (Print): 2347-6710

International Journal of Innovative Research in Science, Engineering and Technology

(An ISO 3297: 2007 Certified Organization)

Vol. 4, Issue 1, January 2015

Common Fixed Point Theorem Governed by Implicit Relation and Property (E. A.)

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ABSTRACT: In order to demonstrate the utility of implicit relation in metric space, we have added common fixed point theorem through this paper. It is a generalized work on pointwise R-weakly commuting and compatible mappings sharing the common property (E. A.). This work extends the results contained in available research work over compatible mappings and as a bi-product we obtain new theorem in metric spaces.

KEYWORDS: **:** Compatible maps, pointwise R-weakly commuting mappings, property (E. A.), implicit relation.

I. INTRODUCTION

The theory of probabilistic metric spaces introduced by Menger [3], where a distribution function was used instead of non negative real number as value of the metric. Sehgal [6] derive the concept of contraction mapping theorems over there. Here it may be noted that the notion of compatible mapping is due to Jungck [2]. This concept has been frequently uses to derive theorems on fixed points. Aamri and Moutawakil [1] introduced property (E. A.) and common property (E. A.), which is a successful and popular generalization of compatible and non compatible mappings in metric space. Their work extended by Imdad et al. [5] in field of semi metric spaces while Kubiaczyk and Sharma [4] developed it in Menger space under strict contractive conditions. The concept of weakly commuting mappings in PM spaces developed by Singh et al. [14]. Kumar and Chugh [15] derived some theorems in metric spaces by using the idea of pointwise R-weakly commutativity. In present paper we utilize these concepts to prove our theorem for six mappings in PM space, which generalize known results of [7] and [9].

II. PRELIMINARIES

A metric is like a function that satisfies the minimal properties we might except of a distance.
We begin with some known definitions.

Definition.2.1. A metric d on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y \in X$:
(i). $d(x, y) \geq 0$ and $d(x, y)= 0$ iff $x = y$,
(ii). $d(x, y) = d(y, x)$, (symmetry)
(iii). $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).
A metric space (X, d) is a set X with a metric d defined on X and has a notion of the distance d(x, y) between every pair of points $x, y \in X$.We can define many different metrics on the same set, but if the metric on X is clear from the context, we refer to X as a metric space and omit explicit mention of the metric d.

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Definition.2.2[10]. Self-maps S and T to be weakly commuting if
d(STx, TSx) ≤ d(Sx, Tx), for ∀ x ∈ X

Definition.2.3[2]. Self-maps S and T to be compatible as a generalization of weakly commuting if
$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$

Definition.2.4[11]. Self-maps S and T are said to be weakly compatible if they commute at their coincidence points. i.e. if Su = Tu then STu = TSu, for u ∈ X.
If mappings are compatible, then they are weakly compatible, but converse is not true.

Definition.2.5[12]. Self-maps S and T are said to be R-weakly commuting if there exits an R such that
d(STx, TSx) ≤ R d(Sx, Tx),

Definition.2.6[12]. S and T are said to be pointwise R-weakly commuting if there exits an R > 0 such that previous condition holds.

R-weakly commutativity is equivalent to commutativity at coincidence points. i.e., S and T are pointwise R-weakly commuting iff they are weakly compatible.

Definition.2.7[1]. S and T satisfy property (E. A.) if there exists a sequence $\{x_n\}$ in X such that
$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for t ∈ X.

Definition.2.8[13]. The mappings A, B, S, T of a metric space (X, d) satisfy a common property (E. A.) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$, for t ∈ X.

III. MAIN RESULT

To prove the fixed point theorem, we follow the idea of a class of implicit functions initiated by Popa [8], because it covers several contractive conditions rather than one.
For this let F_6 be the set of all real-valued continuous functions $F_6(t_1,\dots, t_6)$: $R_+^6 \to R$, non decreasing in the first argument and satisfying the following conditions for all u ∈ (0, 1) :

(F_1). F(u, 0, u, 0, 0, u) > 0
(F_2). F(u, 0, 0, u, u, 0) > 0
(F_3). F(u, u, 0, 0, u, u) ≥ 0

Example: $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a\,t_2 - b\,\frac{t_3t_6}{t_5+t_6+1} - c\,t_4$
(F_1). F(u, 0, u, 0, 0, u) = $\frac{u}{u+1} > 0$
(F_2). Similarly F(u, 0, 0, u, u, 0) > 0
(F_3). F(u, u, 0, 0, u, u) ≥ 0
Where 0 < a ≤ 1, b ≤ 1, 0 < c < 1 and u ∈ (0, 1).

Lemma.3.1. Let (p, hk) and (q, fg) be pointwise R-weakly commuting pairs of self mappings of a metric space (X, d) satisfying following inequalities
(i). p(X) ⊂ fg(X), q(X) ⊂ hk(X),
(ii). Pairs satisfy property (E. A.),
(iii). F[d(px, qy), d(hkx, fgy), d(px, hkx), d(qy, fgy), d(hkx, qy), d(px, fgy)] < 0.
Then pairs (p, hk) and (q, fg) also share the common property (E. A.).

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Proof. Suppose that the pair (p, hk) have the property (E. A.) and $\{x_n\} \in X$ is a sequence such that $\lim_{n\to\infty} px_n = \lim_{n\to\infty} hkx_n = t$, for $t \in X$.

As $p(X) \subset fg(X)$, then there should be $\{y_n\} \in X$ for every $\{x_n\} \in X$ such hat $px_n = fgy_n$
Therefore it is clear that $\lim_{n\to\infty} px_n = \lim_{n\to\infty} fgy_n = t$.

Now we assert that $\lim_{n\to\infty} gy_n = t$.
For this starts from assumption $\lim_{n\to\infty} gy_n \neq t$.

By inequality (iii) we get
$F[d(px_n, qy_n), d(hkx_n, fgy_n), d(px_n, hkx_n), d(qy_n, fgy_n), d(hkx_n, qy_n), d(px_n, fgy_n)] < 0$

Which on making $n \to \infty$
$F[d(t, qy_n), d(t, t), d(t, t), d(qy_n, t), d(t, qy_n), d(t, t)] < 0$
$F[d(t, qy_n), 0, 0, d(qy_n, t), d(t, qy_n), 0] < 0$

Which is contradiction to (F_2). Hence it is clear that $\lim_{n\to\infty} gy_n = t$.
Thus we can declare that pairs (p, hk) and (q, fg) also having the common property (E. A.).

Following theorem generalizes of [9] and [14].

Theorem.3.2. Let (p, hk) and (q, fg) be pointwise R-weakly commuting pairs of self mappings of a metric space (X, d) satisfying following inequalities
(i). $p(X) \subset fg(X)$, $q(X) \subset hk(X)$,
(ii). Having the property (E. A.),
(iii). hk(X) and fg(X) are closed subset of X,
(iv). $F[d(px, qy), d(hkx, fgy), d(px, hkx), d(qy, fgy), d(hkx, qy), d(px, fgy)] < 0$.
For all $x, y \in X$ and $F \in F_6$.
Moreover k commute with p & h and g commute with q & f. If one of the mappings in the pair is continuous then p, q, f, g, h, k have a unique common fixed point in X.

Proof. If pairs (p, hk) and (q, fg) are having the property (E. A.) then from the lemma (3.1) we can say that they also share common property (E. A.).

So that two sequences $\{x_n\}, \{y_n\} \in X$ should be exist such that
$\lim_{n\to\infty} px_n = \lim_{n\to\infty} hkx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} fgy_n = t$, for $t \in X$.

Because hk(X) is closed subset of X thus there $u \in X$, such that $hku = t$.

If $pu \neq t$, then by using (iv) we have
$F[d(pu, qy_n), d(hku, fgy_n), d(pu, hku), d(qy_n, fgy_n), d(hku, qy_n), d(pu, fgy_n)] < 0$

Applying $n \to \infty$
$F[d(pu, t), d(t, t), d(pu, t), d(t, t), d(t, t), d(pu, t)] < 0$
$F[d(pu, t), 0, d(pu, t), 0, 0, d(pu, t)] < 0$

This is contradiction to (F_1) and it is clear that $pu = t$.

Since $p(X) \subset fg(X)$, there exist $v \in X$ such that $pu = fgv = t$.

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If gv ≠ t, then by (iv) we get
F[d(pu, qv), d(hku, fgv), d(pu, hku), d(qv, fgv), d(hku, qv), d(pu, fgv)] < 0

Taking $n \to \infty$
F[d(t, qv), d(t, t), d(t, t), d(qv, t), d(t, qv), d(t, t)] < 0
F[d(t, qv), 0, 0, d(qv, t), d(t, qv), 0] < 0

This is contradiction of (F_2) and therefore gv = t.
If we combine all the results, we get pu = hku = qv = fgv = t.

Because pairs (p, hk) and (q, fg) are pointwise R-weakly commuting, so that
pt = phku = hkpu = hkt
qt = qfgv = fgqv = fgt

If pt ≠ t, then by (iv) we obtain
F[d(pt, qv), d(hkt, fgv), d(pt, hkt), d(qv, fgv), d(hkt, qv), d(pt, fgv)] < 0
F[d(pt, qv), d(pt, fgv), d(pt, pt), d(qv, fgv), d(pt, qv), d(pt, fgv)] < 0

If $n \to \infty$
F[d(pt, t), d(pt, t), d(pt, pt), d(t, t), d(pt, t), d(pt, t)] < 0
F[d(pt, t), d(pt, t), 0, 0, d(pt, t), d(pt, t)] < 0

This is contradiction of (F_3) and it is clear that pt = t and so hkt =t.

At last if qt ≠ t, then from (iv) we have
F[d(pt, qt), d(hkt, fgt), d(pt, hkt), d(qt, fgt), d(hkt, qt), d(pt, fgt)] < 0
F[d(pt, qt), d(hkt, qt), d(pt, hkt), d(qt, qt), d(hkt, qt), d(pt, qt)] < 0

Using $n \to \infty$
F[d(t, qt), d(t, qt), d(t, t), d(qt, qt), d(t, qt), d(t, qt)] < 0
F[d(t, qt), d(t, qt), 0, 0, d(t, qt), d(t, qt)] < 0

This is again a contradiction of (F_3), hence qt = t and similar fgt =t.
It means pt = hkt = qt = fgt = t

By the features of pointwise R-weakly commuting mapping
hkt = ht, fgt = ft, hkt = kht = kt, fgt = gft = gt.
So that pt = ht = kt = qt = ft = gt = t.
Or we can say p, h, k, q, f, g have a common fixed point in X.
This completes the proof.

Uniqueness : Let w is another common fixed point of all define mappings such that t ≠ w, by (iv)
F[d(pt, qw), d(hkt, fgw), d(pt, hkt), d(qw, fgw), d(hkt, qw), d(pt, fgw)] < 0
F[d(t, w), d(t, w), d(t, t), d(w, w), d(t, w), d(t, w)] < 0
F[d(t, w), d(t, w), 0, 0, d(t, w), d(t, w)] < 0

Which is contradiction to (F_3), so that it is clear that t = w.
In this way, we can easily show the existence of an unique common fixed point. This concludes the proof.

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IV. CONCLUSION

In this paper through Theorem 3.2 we introduce the new concept of common fixed point in case of newly defined implicit relation.

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